Reminder

- For a simple Lie algebra / $\mathbb{C}$ $\mathfrak{g}$ we choose a Cartan

$$\mathfrak{h} \subset \mathfrak{g}$$

For $\mathfrak{g} = \mathfrak{sl}_2 = \{ A \in Mat_2(\mathbb{C}) \mid \operatorname{tr} A = 0 \}$.

we can choose

$$\mathfrak{h} = \{ A \in Mat_2(\mathbb{C}) \mid A = \operatorname{diag}(\alpha, -\alpha) \}.$$

- Let $\mathfrak{h} \subset \mathfrak{g}$ via $(\operatorname{ad}\mathfrak{g})$ The $\mathfrak{h}$-weight spaces

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$$

are called the roots

A matrix $\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} = D \in \mathfrak{h}$ acts via $[D, -]$

so

$$D \cdot X = [D, X] = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & -a \end{pmatrix} - \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}$$

$$= \begin{pmatrix} \alpha a & \alpha b \\ -\alpha c & \alpha a \end{pmatrix} - \begin{pmatrix} \alpha a & -\alpha b \\ \alpha c & \alpha a \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2\alpha b \\ -2\alpha c & 0 \end{pmatrix}$$

So we immediately see two eigenvectors

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and $\mathfrak{g} = \mathfrak{sl}_2 = \mathbb{C}e \oplus h \oplus \mathbb{C}f$

$\mathbb{C}e = \mathfrak{g}_\alpha$ for $\alpha(D) = 2d$

$\mathbb{C}f = \mathfrak{g}_{-\alpha}$

$h = \mathfrak{g}_0$

- The set $\Phi = \{\alpha \mid \alpha \text{ is a root}\}$ is called the root system. We have

$$E_0 = \mathbb{Q}\Phi \subset h^*, \quad E = \mathbb{R} \otimes_{\mathbb{Q}} E_0$$

~~For $\alpha \in \Phi$ we define~~

On h we have the Killing form

$$(x\,y) := \mathrm{Tr}(\mathrm{ad}\,x\,\mathrm{ad}\,y)$$

If $C, D \in \mathfrak{sl}_2$ then $\mathfrak{sl}_2$ is 3-dim and wrt the basis $\{e\ h\ f\}$

$$\mathrm{ad}\,D = \begin{pmatrix} 2d & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2d \end{pmatrix} \quad \text{so } (C,D) = 8cd$$

- So E is ~~non-degenerate~~ a Euclidean space define

$$\alpha \in \Phi \quad s_\alpha : \lambda \longmapsto \lambda - 2(\lambda,\alpha)/(\alpha,\alpha)$$

the simple reflections

the Weyl group is the subgroup $W \subset GL(E)$ generated by the s_α

In our case we have

$$s_\alpha : \lambda \longmapsto \lambda - \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}$$

~~But~~ first we figure out that

$$(\lambda,\mu) = \frac{1}{8}\cdot\lambda(h)\mu(h)$$

so we see

$$s_\alpha(\lambda)(h) = \lambda(h) - \frac{2\left(\frac{1}{8}\lambda(h)\alpha(h)\right)}{\frac{1}{8}\alpha(h)\alpha(h)}\alpha(h)$$

$$= \lambda(h) - 2\lambda(h)$$

$$= -\lambda(h).$$

so $s_\alpha : \lambda \longmapsto -\lambda$

and $s_{-\alpha} = s_\alpha$ trivially.

So $W = S_2 = \{\{id, -id\} \subset GL_1(\mathbb{R})$.

- We choose a simple system $\Delta \subset \Phi$ (a basis for E) and define

$$\Phi^+ := \mathbb{R}_+ \cdot \Delta \cap \Phi \quad \text{the positive roots}$$

$$\Phi^- := (\mathbb{R}_{<0} \Delta) \cap \Phi \quad \text{the negative roots.}$$

Choose $\alpha : h \longmapsto 2$

$\{\alpha\} = \Phi^+$, $\{-\alpha\} = \Phi^-$

- The $\mathbb{Z}$-span of $\Phi = \Lambda_r$ root lattice

Define $\alpha^\vee := 2\alpha/(\alpha,\alpha)$ the coroot

$$\Phi^\vee = \{\alpha^\vee : \alpha \in \Phi\}$$

is the ~~coweight~~ coroot system, $\Lambda_r^{co} = \mathbb{Z}\Phi^\vee$ the coroot lattice

$$\Lambda := \left\{ \lambda \in E \mid \langle \lambda \alpha^\vee \rangle = \frac{2(\lambda \alpha^\vee)}{(\alpha^\vee \alpha^\vee)} \in \mathbb{Z} \quad \forall \alpha \in \Delta \right\}.$$

is the weight lattice

For $\alpha : h \longmapsto 2$, $\alpha^\vee = \dfrac{2\alpha}{(\alpha^\vee \alpha^\vee)} = \dfrac{2\alpha}{\frac{1}{8}\alpha(h)\alpha(h)} = 4\alpha$

$\Lambda_r = \mathbb{Z}\alpha \; (= 2\mathbb{Z})$

and we find $\Lambda = \mathbb{Z}$.

- Consider the matrix $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$.

$$C_\Phi = \left(\langle \alpha_i, \alpha_j \rangle \right)$$

the cartan matrix

From this we construct the dynkin diagram:

- ℓ vertices corre Δ
- i joined to j by $\langle \alpha_i \alpha_j \rangle \langle \alpha_j \alpha_i \rangle$ edges.

- If more than one edge and $\langle \alpha_i \alpha_j \rangle < \langle \alpha_j \alpha_i \rangle$ then an arrow from j to i

Trivially for $\underline{sl}_2$ $C = (2)$ and

D: ∘

Constructing infinite dim examples

- We have a complete list of (simple) f.d Lie algebras. To create some inf dim examples take a f.d simple Lie algebra $\mathfrak{g}$ and a $\mathbb{C}$-algebra R:

$$\mathfrak{g} \otimes R \quad \text{with } [x \otimes r, y \otimes s] = [x\,y] \otimes rs$$

will be infinite dimensional

Def $L\mathfrak{g} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ is the loop algebra.

Central extensions

- For various reasons we need a central element in our construction.

If $\underline{a}$ is an abelian Lie algebra and $\mathfrak{g}$ a Lie algebra then $\bar{\mathfrak{g}}$ is called a central extension of $\mathfrak{g}$ by $\underline{a}$ if there exist a s.es

$$0 \longrightarrow \underline{a} \longrightarrow \bar{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0$$

such that $\underline{a} \subset Z(\bar{\mathfrak{g}})$.

This splits as vector spaces, say by

$$\sigma : \mathfrak{g} \longrightarrow \bar{\mathfrak{g}}$$

and we define

$$\beta : \mathfrak{g} \times \mathfrak{g} \longrightarrow \underline{a}$$

$$(x\ y) \longmapsto [\sigma(x)\ \sigma(y)] - \sigma([x\ y])$$

We can check

- $\beta(xy) = -\beta(yx)$
- $\beta(x\ [y\ z]) + \beta(y\ [z\ x]) + \beta(z\ [x\ y]) = 0$

So $\beta \in C^2(\mathfrak{g}; \underline{a})$

- Conversly given $\beta \in C^2(\mathfrak{g}; \underline{a})$ we can define

$$\overline{\mathfrak{g}}_\beta := \mathfrak{g} \oplus \underline{a} \qquad [(g,r),(h,s)] = \left([g,h], \beta(g,h)\right]$$

(In fact we only need β up to $B^2(\mathfrak{g}; \underline{a})$ so central extensions are classified by $H^2(\mathfrak{g}; \underline{a})$)

Central extension of $\mathfrak{L}\mathfrak{g}$

- We have the Killing form $(,)$ on $\mathfrak{g}$ we extend this to a form

$$\mathfrak{L}\mathfrak{g} \times \mathfrak{L}\mathfrak{g} \longrightarrow \mathbb{C}[t, t^{-1}]$$
$$(x \otimes P, y \otimes Q) \longmapsto PQ\,(x,y)$$

Now define

$$\frac{d}{dt} : Lg \longrightarrow Lg \qquad Res : \mathbb{C}[t,t^{-1}] \longrightarrow \mathbb{C}$$

$$x \otimes P \longmapsto x \otimes \frac{dP}{dt} \qquad t^n \longmapsto \delta_{n,-1}$$

and

$$\beta(x\otimes P, y\otimes Q) = Res\left(\frac{d}{dt}x\otimes P,\ y\otimes Q\right)$$

$$= Res\left(x\otimes\frac{dP}{dt}, y\otimes Q\right)$$

$$= (x,y)\ Res\left(\frac{dP}{dt}Q\right).$$

We can check that this is a 2-cocyle $C^2(Lg;\mathbb{C})$.

So we define the affine algebra

$$\tilde{g} := Lg \oplus \mathbb{C}c$$

$$[x\otimes P+\alpha c,\ y\otimes Q+\beta c] = [x,y]\otimes PQ + (x,y)\,Res\left(\frac{dP}{dt}Q\right).$$

We can look at the "roots" of $\tilde{\mathfrak{g}}$, the weights showing up under the action of $\mathfrak{h}$

$\tilde{\mathfrak{g}} = \bigoplus \tilde{\mathfrak{g}}_\alpha$ where $\tilde{\mathfrak{g}}_\alpha = \mathfrak{g}_\alpha \otimes \mathbb{C}[t, t^{-1}]$

since $[h \otimes f, x \otimes P] = [h, x] \otimes P$

$= \alpha(h) x \otimes P$ if $x \in \mathfrak{g}_\alpha$.

In fact $\tilde{\mathfrak{g}}_\alpha$ decompose further, into ~~1-dim weight spaces~~

$\mathfrak{g}_\alpha \otimes t^n$

So we have roots $\tilde{\Phi} = \{(\alpha, n) \mid \alpha \in \Phi \; n \in \mathbb{Z}\}$.

This is not very satisfactory! It is an arbitrary (though attractive) decomposition of $\tilde{\mathfrak{g}}_\alpha$.

Ok lets give up on this for now.

Cartan Matrices and KM-algebras

Def An integer $n \times n$ matrix A is called a Cartan matrix if

(1) $a_{ii} = 2$ for all i

(2) $a_{ij} \leq 0$ if $i \neq j$

(3) $a_{ij} = 0 \iff a_{ji} = 0$

(4) $\det A \neq 0$ and all principal minors are positive.

$$\left\{\begin{matrix}\text{Cartan}\\ \text{Matrices}\end{matrix}\right\} \xleftrightarrow{1:1} \left\{\begin{matrix}\text{Dynkin}\\ \text{diagrams}\end{matrix}\right\}$$

$$\swarrow^{1:1} \qquad \searrow^{1:1}$$

$$\left\{\begin{matrix}\text{f.d. simple}\\ \text{Lie algebras}\end{matrix}\right\}.$$

Def A generalised Cartan matrix is an integer matrix obeying (1) – (3).

We want to construct a Lie algebra $g(A)$ for each matrix A. We work backwards:

- Fix a complex vector space and linearly ind subsets

$$\{\alpha_1 \dots \alpha_n\} \subset h^*$$

$$\{h_1 \dots h_n\} \subset h$$

such that $\alpha_i(h_j) = a_{ij}$

Lemma: $\dim h \geq 2n - \operatorname{rank} A$

- Fix h with minimal dimension

Def $g(A)$ to be the Lie algebra generated by symbols

$$e_1 \dots e_n, f_1 \dots f_n, h$$

with rels

$[h, h'] = 0$ for $h, h' \in h$

$[e_i, f_j] = \delta_{ij} h_i$

$[h, e_i] = \alpha(h) e_i$ and $[h, f_i] = -\alpha(h) f_i$

There is also a unique maximal ideal that intersects $\mathfrak{h}$ trivially

$$(\mathrm{ad}\, e_i)^{1-a_{ij}} e_j = 0 \qquad (\mathrm{ad}\, f_i)^{1-a_{ij}} f_j = 0 \qquad i \neq j.$$

Prop The center is

$$Z(\mathfrak{g}(A)) = \{h \in \mathfrak{h} \mid \alpha_i(h) = 0 \ \forall i\}.$$

Theorem A generalised Cartan matrix is one of the following:

(Finite) $\det A \neq 0$, principal minors are +ve

(Affine) $\operatorname{corank} A = 1$, principal minors +ve

(Indefinite) others.

(Finite) are the cartan matrices and $\mathfrak{g}(A)$ is a simple fd Lie algebra.

(Affine) can also be classified by extended Dynkin diagrams

Prop If A is of Affine type then $\mathfrak{g}(A)$ has a 1d center.

proof $\dim \mathfrak{h} = 2n - \operatorname{rank} A = n+1$ so the intersection of the kernels of α_i must be 1d.

- We can now repeat all our usual constructions

$\Delta = \{\alpha_1 - \alpha_n\}$ the simple roots

we have $g(A) = \bigoplus_{\alpha \in \Phi(A)} g(A)_\alpha$ the roots

and $\Lambda_r = \mathbb{Z}\Phi(A)$ the root lattice and reflections

$$s_{\alpha_i} : \lambda \longmapsto \lambda - \lambda(h_i)\alpha_i$$

In fact, from a finite Cartan matrix A we can produce an affine Cartan matrix $(n+1 \times n+1)$ $\hat{A}$ by "extending".

$$\text{eg } A_1 = (2) \rightsquigarrow \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} = \hat{A}_1$$

$$C_2 = \begin{pmatrix} 2 & -1 \\ -2 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix} = \hat{C}_2$$

We have added a zero ^{4h} ~~and~~ row and column, hence there should be a 0^{th} simple root!

More generally: two algebras

L_1, L_2. with L_2 acting on L_1 by derivations

$L_1 \rtimes L_2 \cong L_1 \oplus L_2$ as v.sp.

and bracket

$$[a+x, b+y] = [a,b] + [x,y] + x \cdot b - y \cdot a$$

~~Clearly $\mathrm{Der}\,\mathbb{C}[t,t^{-1}]$~~ take $d = t\frac{d}{dt} \in \mathrm{Der}\left(\mathbb{C}[t,t^{-1}]\right)$

We can extend d to $\overline{\mathfrak{g}}$ by

$$d\left(x \otimes P + \mu c\right) = x \otimes dP$$

and define

$$\hat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}c \rtimes \mathbb{C}d$$

Theorem $\hat{\mathfrak{g}} \cong \mathfrak{g}(\hat{A})$

Our "cartan" is now

$$\hat{h} = h \oplus \mathbb{C}c \oplus \mathbb{C}d$$

We can ask how d acts on ~~weig~~ our root spaces?

$$\bar{g}_\alpha = \bigoplus_{n \in \mathbb{Z}} t^n g_\alpha \otimes t^n$$

say $x \in g_\alpha$ then

$$[d, x \otimes t^n] = d(x \otimes t^n)$$

$$= n\, x \otimes t^n$$

so d splits our eigenspaces!

Def $\hat{g} = Lg \oplus \mathbb{C}c \oplus \mathbb{C}d$

$$[x \otimes t^m + \lambda c + \mu d,\ y \otimes t^n + \lambda' c + \mu' d]$$

$$= [x\, y] \otimes t^{m+n} + \mu n\, y \otimes t^n - \mu' m t^m \otimes x$$

$$+ m \delta_{m+n,0} (x, y)\, c$$

Type B_2/C_2

Let $\mathfrak{g} = \mathfrak{sp}_4 = \{X \in \mathrm{Mat}_4(\mathbb{C}) \mid X^T J + J X = 0\}$

where $J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$

A convenient basis is

$h_1 = \mathrm{diag}(1\ 0\ -1\ 0)$ $h_2 = \mathrm{diag}(0\ 1\ 0\ -1)$

$$x_1 = \begin{pmatrix} 0 & 1 & & \\ 0 & 0 & & \\ & & 0 & 0 \\ & & -1 & 0 \end{pmatrix} \qquad x_2 = \begin{pmatrix} 0 & 0 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 0 & 0 \end{pmatrix}$$

$$a_1 = \begin{pmatrix} & & 0 & 1 \\ & & 1 & 0 \\ & & & \\ & & & \end{pmatrix} \qquad a_2 = \begin{pmatrix} & & & \\ 0 & 1 & & \\ 1 & 0 & & \\ & & & \end{pmatrix}$$

$$b_1 = \begin{pmatrix} & & 1 & 0 \\ & & 0 & 0 \\ & & & \\ & & & \end{pmatrix} \qquad b_2 = \begin{pmatrix} & & & \\ 1 & 0 & & \\ 0 & 0 & & \\ & & & \end{pmatrix}$$

$$c_1 = \begin{pmatrix} & & 0 & 0 \\ & & 0 & 1 \\ & & & \\ & & & \end{pmatrix} \qquad c_2 = \begin{pmatrix} & & & \\ & & & \\ 0 & 0 & & \\ 0 & 1 & & \end{pmatrix}$$

The adjoint action is thus given by

$$[h_1, x_1] = x_1 \qquad [h_2, x_1] = -x_1$$

$$[h_1, x_2] = -x_2 \qquad [h_2, x_2] = x_2$$

$$[h_1, a_1] = a_1 \qquad [h_2, a_1] = a_1$$

$$[h_1, b_1] = 2b_1 \qquad [h_2, b_1] = 0$$

$$[h_1, c_1] = 0 \qquad [h_2, c_1] = 2c_1$$

$$[h_1, a_2] = -a_2 \qquad [h_2, a_2] = -a_2$$

$$[h_1, b_2] = -2b_2 \qquad [h_2, b_2] = 0$$

$$[h_1, c_2] = 0 \qquad [h_2, c_2] = -2c_2$$

By choosing a nice basis we have also found our weight spaces

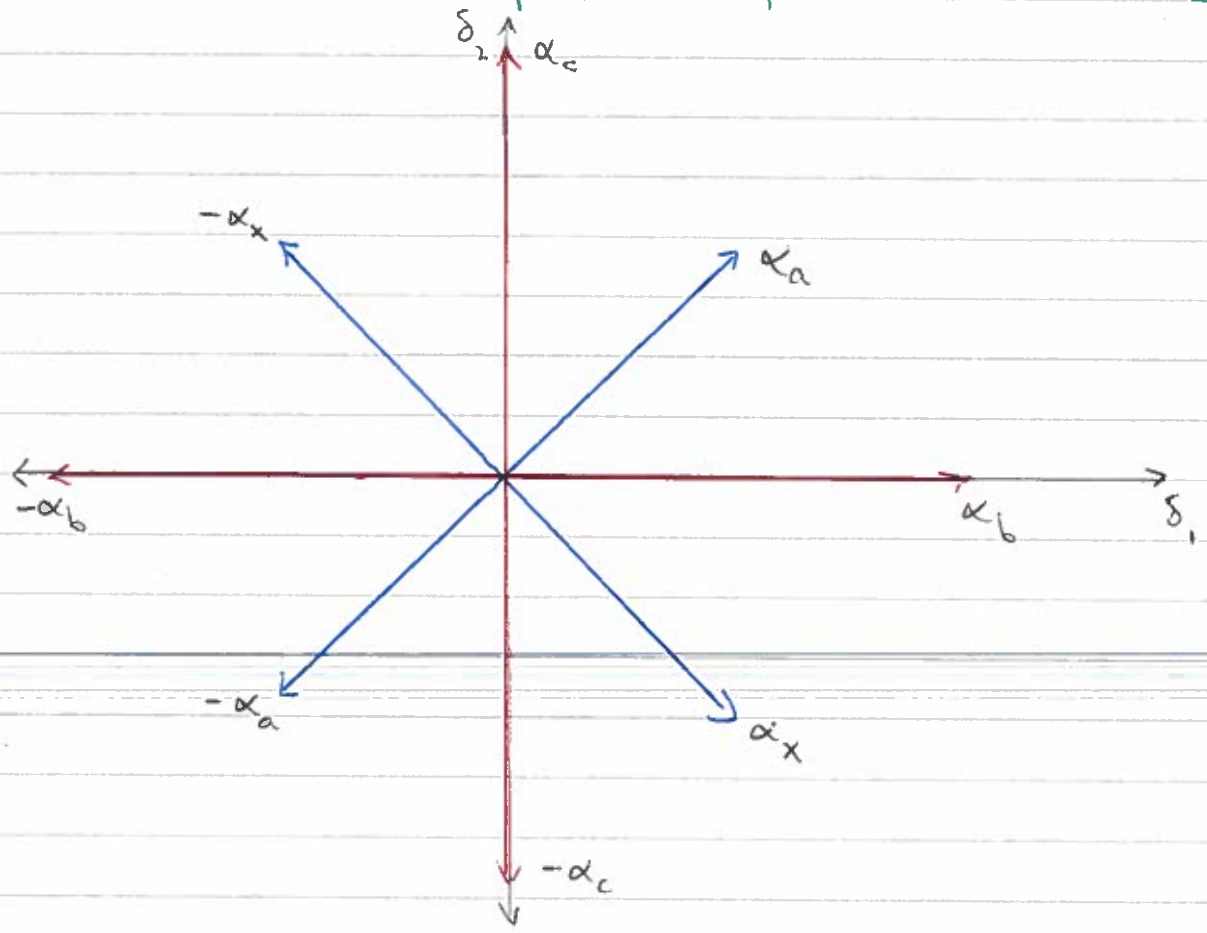
The roots $\in \mathfrak{h}^* = \mathbb{C}\{h_1, h_2\}^*$ are

α_x : $h_1 \longmapsto 1$ $h_2 \longmapsto -1$
α_a : $h_1 \longmapsto 1$ $h_2 \longmapsto 1$
α_b : $h_1 \longmapsto 2$ $h_2 \longmapsto 0$
α_c : $h_1 \longmapsto 0$ $h_2 \longmapsto 2$

and their negatives!

So we also chose our labels intelligently!

We can draw a picture of our roots: in E



Here $\{\delta_1, \delta_2\}$ is the dual basis to $\{h_1, h_2\}$.

So

$$\mathfrak{sp}_4 = \underbrace{\mathbb{C}x_1 \oplus \mathbb{C}a_1 \oplus \mathbb{C}b_1 \oplus \mathbb{C}c_1}_{\mathfrak{n}^+} \oplus \underbrace{\mathbb{C}h_1 \oplus \mathbb{C}h_2}_{\mathfrak{h}} \oplus \underbrace{\mathbb{C}x_2 \oplus \mathbb{C}a_2 \oplus \mathbb{C}b_2 \oplus \mathbb{C}c_2}_{\mathfrak{n}}$$

If we choose say $\Delta = \{\alpha_x, \alpha_c\}$.

(then $\Phi^+ = \{\alpha_x, \alpha_b, \alpha_a, \alpha_c\}$.)

We can now calculate the Killing form on $\mathfrak{h}$:

$$(h, h_1) = \mathrm{Tr}(\mathrm{ad}\, h, \mathrm{ad}\, h_1)$$

$$= \mathrm{Tr}\begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 2 & 0 & & & \\ & & 0 & 0 & & & \\ & & & & -1 & & \\ & & & & & -1 & \\ & & & & & -2 & 0 \end{pmatrix}^2$$

$$= 12$$

similarly

$$(h_2, h_2) = 12 \quad \text{and} \quad (h_1, h_2) = 0$$

Hence $\delta_i = \left(\frac{1}{12} h_i, -\right)$ and transfering the form across to $\mathfrak{h}^*$:

$$(\delta_i, \delta_j) = \frac{1}{12} \delta_{ij}$$

(sorry confusing notation!).

It now follows from our picture that $W = D_4$!

Next & let us calculate the cartan matrix C. Set $\alpha_1 = \alpha_x$ and $\alpha_2 = \alpha_c$ then

$$\langle \alpha_1 \alpha_1 \rangle = 2\frac{(\alpha_1 \alpha_1)}{(\alpha_1 \alpha_1)} = 2 = (\alpha_2 \alpha_2)$$

$$\langle \alpha_1 \alpha_2 \rangle = 2\frac{(\alpha_1 \alpha_2)}{(\alpha_2 \alpha_2)} = 2\frac{(\delta_1 - \delta_2, 2\delta_2)}{(2\delta_2\ 2\delta_2)}$$

$$= \frac{-\frac{1}{3}}{\frac{1}{3}} = -1$$

$$\langle \alpha_2 \alpha_1 \rangle = 2\frac{(\alpha_2 \alpha_1)}{(\alpha_1 \alpha_1)} = 2\frac{(2\delta_2\ \delta_1 - \delta_2)}{(\delta_1 - \delta_2, \delta_1 - \delta_2)}$$

$$= \frac{-\frac{1}{3}}{\frac{1}{6}} = -2$$

So $$C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$

and the Dynkin diagram is

The Affine root system.

We will investigate the root system attached to $\hat{\mathfrak{g}}$ and in the examples $\widehat{\mathfrak{sl}}_2$ $\widehat{\mathfrak{sp}}_4$ show that we get back the affine cartan matrices described earlier.

Recall that $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$ has non-zero weight spaces

$$\hat{\mathfrak{g}}_\alpha = \mathfrak{g}_\alpha \otimes \mathbb{C}t^n$$

for $\alpha \in \Phi \cup \{0\}$ and $n \in \mathbb{Z}$ (but α and n are not both allowed to be zero.

Recall $\Lambda_r = \mathbb{Z}\Phi$, the root lattice for $\mathfrak{g}$. Define an ordering $\lambda \geq \mu$ for $\lambda, \mu \in \Lambda_r$, if

$$\lambda - \mu \in \mathbb{Z}_{\geq 0}\Delta \quad (\text{a +ve sum of simple ~~weights~~ roots}).$$

In Φ, there is a unique maximal root with respect to this order. Say θ.

In $\mathfrak{sl}_2$ $\Phi = \{-\alpha, \alpha\}$ with $\Delta = \{\alpha\}$ so $\theta = \alpha$.

In $\mathfrak{sp}_4$ $\Phi = \{\pm\alpha_c, \pm\alpha_a, \pm\alpha_b, \pm\alpha_x\}$ with $\Delta = \{\alpha_x, \alpha_c\}$

Thus we have $\alpha_a = \alpha_x + \alpha_c$, $\alpha_b = 2\alpha_x + \alpha_c$ and $\alpha_b - \alpha_a = \alpha_x$ so $\theta = \alpha_b$.

To make things a little simpler, normalise the Killing form so that $(\theta,\theta)=2$.

For $\underline{sl_2}$ we define on $\mathfrak{h}=\mathbb{C}h$ the form

$$(a\ b) = K\cdot \mathrm{Tr}(\mathrm{ad}\,a\cdot \mathrm{ad}\,b)$$

for some $K\in\mathbb{C}$ so that $(\theta\,\theta)=(\alpha\,\alpha)=2$. Lets find K. Denote by $D:\mathfrak{h}\longrightarrow\mathfrak{h}^*$ the isomorphism induced by $(-,-)$.

$(\frac{1}{4K}h, h)=2$ so $D(\frac{1}{4K}h)=\alpha$ thus

$$(\alpha\,\alpha) = (\tfrac{1}{4K}h, \tfrac{1}{4K}h)$$

$$= \frac{1}{16K^2}(h,h)$$

$$= \frac{K}{16K^2}\mathrm{Tr}(\mathrm{ad}\,h\cdot \mathrm{ad}\,h)$$

$$= \frac{1}{2K}$$

so we take $K=1/4$.
ie $(h\,h)=2$ and $(\alpha\,\alpha)=2$.

For $\underline{sp_4}$ again define on $\mathfrak{h}=\mathbb{C}\{h_1,h_2\}$ the form
$(a\,b) = K\cdot\mathrm{Tr}(\mathrm{ad}\,a\cdot\mathrm{ad}\,b)$

Now $\alpha_b(h_i)=\delta_{ii}\,2$ and $(\frac{1}{6K}h_1,h_i)=\delta_{ii}\,2$
so $\alpha_b = D(\frac{1}{6K}h_1)$.

Hence $(\theta\theta) = (\alpha_0 \alpha_0)$

$$= \left(\frac{1}{6K} h_1, \frac{1}{6K} h_1\right)$$

$$= \frac{K}{36K^2} \mathrm{Tr}\left(\mathrm{ad}\, h_1 \cdot \mathrm{ad}\, h_1\right)$$

$$= \frac{1}{3K}$$

So $K = 1/6$.

Hence ~~$(h_i h_j) = 2\delta_{ij}$ and~~

~~Note~~ We can now extend this form $(\cdot,\cdot)$ on $\mathfrak{h}$ to $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$ by

$(h, c) = 0$ $\quad (c\,c) = 0$ $\quad (c\,d) = 1$

$(h, d) = 0$ $\quad (d\,d) = 0$

not pos def though

Note that this is still a nondegenerate form (~~exten~~ consider the elements $c \pm d$) hence we still have an isomorphism

$$\nu : \hat{\mathfrak{h}} \longrightarrow \hat{\mathfrak{h}}^* .$$

If we denote $\delta \in \hat{\mathfrak{h}}^*$ the functional

$$\delta(d) = 1 \quad \text{and} \quad \delta(c) = \delta(h) = 0$$

then we see the affine root system is

$$\hat{\Phi} = \{m\delta + \alpha\}_{m \in \mathbb{Z}, \alpha \in \Phi} \cup \{m\delta\}_{m \in \mathbb{Z} \setminus \{0\}}$$

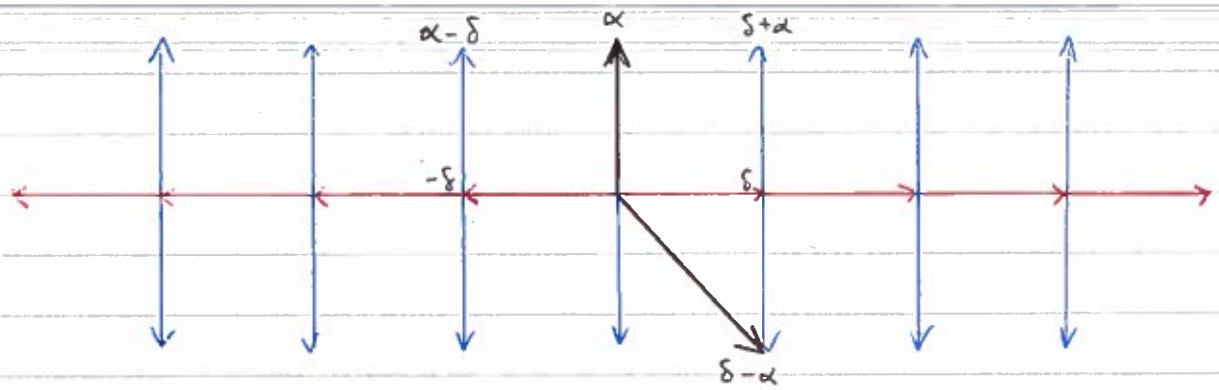
So it is made up of infinitely many "layers" which are just copies of Φ and then "imaginary" roots $n\delta$ going between them.

A simple system is

$$\hat{\Delta} = \Delta \cup \{\delta - \theta\}$$

where Δ is a simple system for Φ.

For $\mathfrak{sl}_2$ we can draw this



$$\hat{\Delta} = \{\alpha, \delta - \alpha\}$$

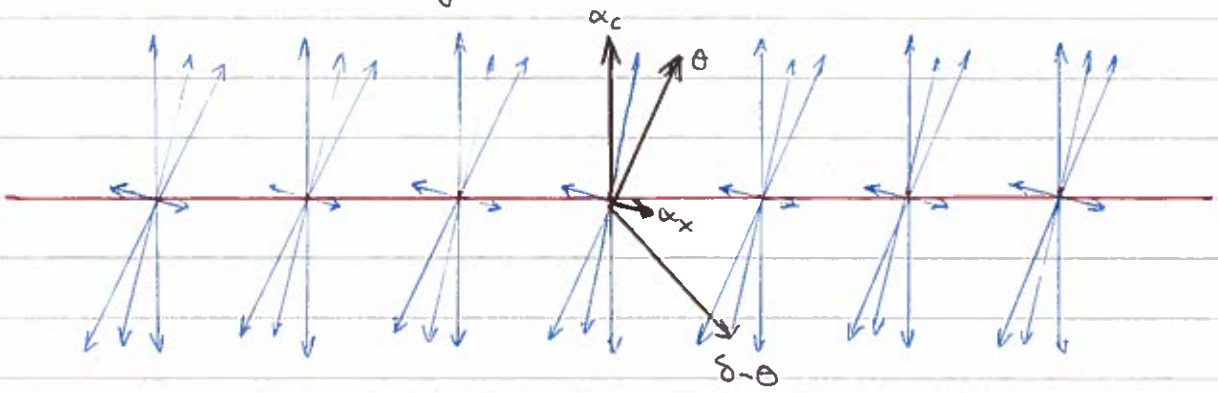
But we should note that this isn't a very good picture since

$$(\delta, \delta) = 0 \quad (\delta - \alpha, \delta - \alpha) = (\alpha, \alpha) = 2$$

and $(\alpha, \delta - \alpha) = -2$

so α and $\delta - \alpha$ are somehow ~~perpendicular~~ seperated by 180°.

For $\underline{sp_4}$ we can draw a similar picture in 3d (though artistically much worse).



We can define the coroot $\lambda^\vee \in \hat{h}$ for a root $\lambda \in \hat{\Phi}$ if $(\lambda\lambda) \neq 0$ by

$$\lambda^\vee = \frac{2}{(\lambda\lambda)} \nu^{-1}(\lambda).$$

In this case we get the affine, or extended, cartan matrices by

$$\hat{A} = (a_{ij})_{ij=0}^{\wedge} \quad a_{ij} = \alpha_i(\alpha_j^\vee)$$

for $\hat{\Delta} = \{\alpha_0\ \alpha_1 \text{---} \alpha_n\}$ with $\Delta = \{\alpha_1 \text{---} \alpha_n\}$. In this case the finite cartan matrix A is the submatrix $A = (a_{ij})_{ij=1}^{\wedge}$.

For $\underline{sl_2}$: $\alpha_1 = \alpha$, $\alpha_0 = \delta - \alpha$, as discussed before $\nu(h) = \alpha$ and we have $\nu(c) = \delta$ so

$$\alpha_1^\vee = h \quad \text{and} \quad \alpha_0^\vee = c - h$$