The noncommutative Chern character | April 23, 2018 |

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The NC Chern character is a generalization of the classical Chern character $Ch(V) \in \prod_{n\geq 0} H^{2n}_{DR}(X)$ of a vector bundle V over X. Before discussing this, we'll describe an application of the degree 0 part of this to Cherednik algebras.

Morita equivalence of Cherednik algebras [Berest, Etingof, Ginzburg '04]

- Let $W = S_n$, $\mathfrak{h}$ = reflection representation, $c \in \mathbb{C}$
- $H_c := \dfrac{\mathbb{C}[\mathfrak{h}] * \mathbb{C}[\mathfrak{h}^*] * \mathbb{C}W}{\begin{array}{c} wxw^{-1} = w(x), \quad wyw^{-1} = w(y) \qquad x, y \in \mathfrak{h}, \mathfrak{h}^*, w \in W \\ yx - xy = \langle y, x\rangle - \sum_{\alpha \in R_+} c \langle y, \alpha\rangle\langle \alpha^\vee, x\rangle \end{array}}$

$B_c := eH_ce$

- Rmk: (1) $H_0 = \mathcal{D}(\mathfrak{h}) \rtimes W$, and $B_0 = \mathcal{D}(\mathfrak{h})^W$, where $e = \frac{1}{n!}\sum \sigma$

(2) The H_c are interesting algebras for many reasons, that we won't mention.

Q: When are B_c, $B_{c'}$ isomorphic or Morita equivalent?

Thm [BEG] If $c \notin \overline{\mathbb{Q}}$, then

(a) H_c and $H_{c'}$ are isomorphic iff $c = \pm c'$

(b) " " " Morita equivalent iff $c \pm c' \in \mathbb{Z}$

(c) B_c and $B_{c'}$ are isomorphic iff $c = c'$ or $c = -c' - 1$,

(d) " " " Morita equivalent iff $c \pm c' \in \mathbb{Z}$.

We'll focus on (c).

- For an algebra A, $K_0(A) := \mathbb{Z}\langle \text{projective } A\text{-modules}\rangle / [P] + [P'] = [P \oplus P']$
- $\chi : K_0(A) \longrightarrow HH_0(A), = A/[A, A]$

$M_n(A)e \longmapsto tr(e)$, where $e^2 = e$

The NC Chern character

<u>Facts</u>: (1) $K_0(\mathbb{C}W) \cong \bigoplus_{\tau \in \Gamma} \mathbb{Z}\{\tau\}$ where $\Gamma = \{\text{irreps of } W\}$

(2) $\mathbb{C}W \hookrightarrow H_c$ induces $K_0(\mathbb{C}W) \xrightarrow{\sim} K_0(H_c)$, $\tau \mapsto P_\tau := H_c \otimes_{\mathbb{C}W} \tau$

(3) $eH_c \otimes_{H_c} - : H_c\text{-mod} \to B_c\text{-mod}$ an equivalence

(4) $\dim(HH_0(B_c)) = 1$, $Tr_{B_c}(1) \neq 0$ for generic $c \in \mathbb{C}$

(where $Tr(-) : A \to A/[-,-]$ is the projection)

<u>Pf sketch</u>: (i) $HH_0(B_c)$ coherent sheaf over $\mathbb{C}[c]$,

(ii) B_c has a f.d. rep. for infinitely many c, ~~and~~ so $Tr(1) \neq 0$ for these c.

(5) <u>key point</u>: $\chi_{B_c}(eP_\tau) = G_\tau(nc) \cdot Tr_{B_c}(1)$, where

$$G_\tau(x) = \dim(\tau)\left(1 - \frac{F_\tau(x)}{F_{triv}(x)}\right), \text{ and}$$

$$F_\tau(x) = \prod_{(i,j) \in \text{Young diagram for } \tau} (x + j - i)$$

In other words, $K_0(B_c)$ and $HH_0(B_c)$ "don't depend on c" (generically), but $\chi : K_0 \to HH_0$ <u>does</u> depend on c.

<u>Sketch of proof of Thm (c)</u>:

Suppose $\varphi : B_c \xrightarrow{\sim} B_{c'}$. Then

(i) $K_0(\varphi)([eP_\tau(c)]) = \sum_\sigma m_{\tau\sigma}[eP_\sigma(c')]$ for some $[m_{\tau\sigma}] \in GL(\mathbb{Z})$

(ii)
$$\begin{tikzcd} K_0(B_c) \arrow[r, "\chi"] \arrow[d, "K_0(\varphi)"] & HH_0(B_c) \arrow[d, "HH_0(\varphi)"] \\ K_0(B_{c'}) \arrow[r, "\chi"] & HH_0(B_{c'}) \end{tikzcd}$$
commutes, and $HH_0(\varphi)(Tr_{B_c}(1)) = Tr_{B_{c'}}(1)$

$\overset{(i),(ii),(5)}{\Longrightarrow} \left(\sum_\sigma m_{\tau\sigma} G_\sigma(nc') - G_\tau(nc)\right) Tr_{B_{c'}}(1) = 0$

$\overset{(4)}{\Longrightarrow} \left(\sum_\sigma m_{\tau\sigma} G_\sigma(nc') - G_\tau(nc)\right) = 0 \xrightarrow{\text{magic}} c = c' \text{ or } c = -c' - 1.$

Reference: Loday, Cyclic Homology

The classical Chern character (affine case)

Let A be a commutative k-algebra, M a projective module.

Def: (1) A connection on M is a k-linear map

$$\nabla : M \otimes_A \Omega^n_A \longrightarrow M \otimes_A \Omega^{n+1}_A \qquad \text{satisfying}$$

$$(*) \quad \nabla(x\,w) = \nabla(x)\,w + (-1)^n x\,dw \qquad \text{for } x \in M\otimes\Omega^n,\ w \in \Omega^p$$

(2) The curvature of ∇ is the restriction of ∇^2 to M:

$$R : M \to M \otimes_A \Omega^2.$$

Explicitly, $R(X,Y)(m) = (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})(m)$ for vector fields X, Y.

Rmk: If M is projective, we can view R as an element of $\mathrm{End}_A(M) \otimes_A \Omega^2_A$ (important: R is A-linear), and there is a trace map $\mathrm{End}_A(M) \to A$. (Also, connections exist.)

(8.1.6) Thm: Let $ch(M) := tr(\exp(R)) \in \prod_{n\geq 0} \Omega^{2n}_A$.
The class of $ch(M)$ ~~This~~ is independent of ∇ and is a closed form. Also,

(i) $ch(M \oplus M') = ch(M) + ch(M')$

(ii) $ch(M \otimes M') = ch(M)\, ch(M')$,

i.e. $ch : K_0(A) \longrightarrow H^{ev}_{DR}(A)$ is a ring homomorphism

(8.1.8.1) Explicitly: $\boxed{ch(A^{\oplus r}e) = \frac{1}{n!}\text{ class of } tr(e\,de\cdots de) \in \Omega^{2n}_A}$

Simplest NC version

- Let $e \in M_r(A)$ be an idempotent, and assume $\mathbb{Q} \subset k$.
- Let $t : A^{\otimes n+1} \to A^{\otimes n+1}$, $t(a_0, \dots, a_n) = (-1)^n (a_n, a_0, \dots, a_{n-1})$

$$C^\lambda_n(A) := A^{\otimes n+1}/1-t,$$

$$b : C^\lambda_{n+1} \to C^\lambda_n, \quad b(a_0, \dots, a_n) = (-1)^n(a_n a_0, a_1, \dots, a_{n-1}) + \sum_{i=0}^{n-1}(-1)^i(a_0, \dots, a_i a_{i+1}, \dots)$$

$H^\lambda_n(A)$ = homology of this complex.

(8.3.2) Thm: $ch^{\lambda}_{0,n}: K_0(A) \to H^{\lambda}_{2n}(A)$ given by
$\boxed{ch^{\lambda}_{0,n}([e]) = tr((-1)^n e^{\otimes 2n+1})}$ is well-defined and functorial in A.

(here $[e]$ is the class of $(A^{\oplus r})e$ for an idempotent $e \in M_r(A)$)

(2.1.5) Rmk: (1) Since $\mathbb{Q} \subset k$, $H^{\lambda}_{2n}(A) \cong HC_{2n}(A)$

(8.3.4) (2) A very similar formula defines $ch_{0,n}: K_0(A) \to HC_{2n}(A)$ without assuming $\mathbb{Q} \subset k$. (some rescaling is involved)

(8.3.8) (3) Another similar formula defines $ch_0^-: K_0(A) \to HC_0^-(A) = HC_0^{per}(A)$ (I think, but I didn't understand this formula...)

(8.3.9) Prop: $K_0(A) \to HC_0^{per}(A) \to H^{ev}_{DR}(A)$ is the classical Chern character, up to rescaling (if A commutative and $\mathbb{Q} \subset k$).

Rmk: This is plausible because the two formulas in red boxes are so similar...