WORKING SEMINAR ON REPRESENTATION THEORY OF FINITE DIMENSIONAL ALGEBRAS

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Teaser: Short overview by Andrew T. Carroll.

Time: The working seminar will meet 9–11 every Tuesday, for 10 sessions between Tuesday 6th October and Tuesday 8th December 2015 inclusive.

Audience: All PhD students, postdocs and faculty members are invited to attend.

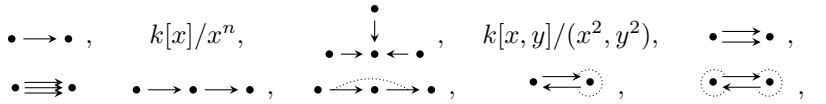
Credit & Assessment: If you are a first year PhD student and wish to take this seminar for credit please email both m.wemyss@ed.ac.uk and m.kalck@ed.ac.uk *as soon as possible* to register. Students taking the seminar for credit will need to be assessed giving a short talk on the material, and in addition will be required to TeX some of the course notes.

Format: We will work through the schedule below, with participants presenting and discussing the material as we go. The first three weeks will be assigned after the first seminar, with further material assigned on a rolling basis. We will progress rather quickly and therefore expect participants to prepare before coming to the sessions, using the given references. This will allow us to spend around half the time discussing questions and doing examples.

Webpage: http://hodge.maths.ed.ac.uk/tiki/FDalgSeminar.

The draft syllabus of the seminar is as follows. References are to the ArXiv as much as possible, so that all participants can access them easily.

- **Session 1: Basics of Quivers, Representations and Examples**.
  - Algebras, Quivers, Modules and Representations.
  - Krull–Schmidt property and Jordan-Hölder filtrations.
  - List of key examples and their properties, including:



References:

(1) Chapter I and II of: I. Assem, D. Simson and A. Skowronski, *Elements of the Representation Theory of Associative Algebras Vol. 1. Techniques of representation theory.* London Mathematical Society Student Texts, **65**. Cambridge University Press, Cambridge, 2006. x+458 pp.

(2) Chapter I and II of: R. Schiffler, *Quiver representations.* CMS Books in Mathematics/Ouvrages de Mathmatiques de la SMC. Springer, Cham, 2014. xii+230 pp. (Springer Link)

(3) Section 6 of: M. Wemyss, *Lectures on Noncommutative Resolutions*, ArXiv:1210.2564.

- **Session 2: AR duality and AR theory**.
  - Projective and injective modules, simple modules, Ext groups.
  - Classes of simple modules form a basis of the Grothendieck group $K_0(\operatorname{mod} A)$ of the category of finite dimensional modules mod A of a finite dimensional algebra A. If A has finite global dimension, classes of indecomposable projectives yield another basis, see eg Assem, Simson, Skowronski, Section III.3.
  - Definition of Auslander–Reiten (=AR) translation τ and statement and proof of Auslander–Reiten (=AR) duality (aka AR formula).
  - AR duality yields existence of AR sequences (aka almost split sequences).

References:

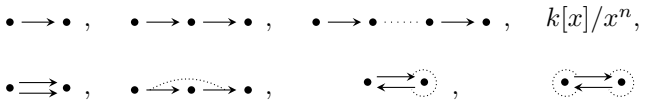
For homological algebra background, e.g. projectives, simples, injectives:

(1) Sections 2.1, 2.2, 2.4 in Schiffler *Quiver representations* (Springer Link).
(2) Chapters 2 & 3 in: Weibel *An introduction to homological algebra*, 1994.
(3) Sections I.4 – I.7; III.1 – III.5 in: Hilton, Stammbach, *A course in homological algebra*, 1971.
(4) Sections III.2 & III.3 in: Assem, Simson, Skowronski, *Elements of the representation theory of associative algebras I*, 2006.

For AR theory:

(5) Section 7 in Schiffler *Quiver representations* (Springer Link).
(6) Sections IV.1 – IV.3 & IV.6 in: Assem, Simson, Skowronski, *Elements of the representation theory of associative algebras I*, 2006.
(7) Chapter V in: Auslander, Reiten, Smalo, *Representation theory of Artin algebras*

- **Session 3: AR quivers, Auslander Algebras and Examples**.
  - Alternative characterisations of AR sequences.
  - First examples (e.g. AR sequence between $k[x]/x^n$ modules.)
  - Definition of AR quivers and alternative descriptions.
  - Definition of Auslander algebras and their properties ($\to$ Auslander correspondence).
  - Examples of AR quivers, including:



References:

(1) Section 3 and 7 in Schiffler's book *Quiver representations*, (Springer Link).
(2) Section VI.5 in: Auslander, Reiten, Smalo, *Representation theory of Artin algebras* (for Auslander algebras)
(3) Section IV.4 in: Assem, Simson, Skowronski, *Elements of the representation theory of associative algebras I*, 2006.

- **Session 4: Morita Theory**.
  - Direct quiver proof that $\operatorname{mod} M_2(k) \simeq \operatorname{mod} k$, see e.g. 6.8(2), 6.12(2), 6.18, 6.20(2) and 6.25 in Reference (1) below.
  - Basics of Morita theory: definition of progenerators, leading to statement and proof of Morita theorems.
  - First examples, including basic algebras, and $\mathbb{C}G$ in the case G is abelian and non-abelian.

References:

(1) Section 6 of: M. Wemyss, *Lectures on Noncommutative Resolutions*, ArXiv:1210.2564.
(2) Chapter II of: H. Bass, *Algebraic K-theory*, Benjamin, 1968.

- **Session 5: Tilting and τ-tilting**.
  - Torsion pairs.
  - Tilting modules, and applications to torsion pairs. Statement of relevance to derived categories. Examples of tilting modules in known examples.
  - Bongartz completion, and Riedtmann–Schofield mutation of tilting modules.
  - τ-tilting modules, and examples.

References:

(1) Chapter VI of: I. Assem, D. Simson and A. Skowronski, *Elements of the Representation Theory of Associative Algebras Vol. 1*, 2006.
(2) K. Bongartz, *Tilted algebras*, Representations of Algebras (Puebla, 1980), Lecture Notes in Math., vol. 903, Springer-Verlag, Berlin-New York, 1981, pp. 26–38.
(3) C. Riedtmann and A. Schofield, *On a simplicial complex associated with tilting modules*, Comment. Math. Helv. **66** (1991), no. 1, 70–78.
(4) T. Adachi, O. Iyama and I. Reiten, *τ-tilting theory*, ArXiv:1210.1036.
(5) O. Iyama and I. Reiten, *Introduction to τ-tilting theory*, ArXiv:1312.7678

- **Session 6: More Homological Algebra and Special Classes of Algebras**.
  - Projective dimension, global dimension, representation dimension, dominant dimension, injective dimension.
  - Special classes of algebras, including some of: quasi-Frobenius (i.e. projectives coincide with injectives), Frobenius, symmetric, Iwanaga–Gorenstein, Nakayama (each module is uniserial), gentle algebras.
  - Quasi-hereditary algebras and their properties.

  References:
  (1) Assem, Simson, Skowronski, *Elements of the representation theory of associative algebras I*, 2006. (for Nakayama (Ch. V), gentle (Ch IX.6)) (Appendix A.4 for projective dimension, global dimension, injective dimension)
  (2) C. Ringel, *Iyama's finiteness theorem via strongly quasi-hereditary algebras* available here (11 pages) (Section 4 for (strongly) quasi-hereditary algebras; page 6 for representation dimension)
  (3) Sections IV.3 (for selfinjective & symmetric algebras), VI.5 (for dominant dimension) in: Auslander, Reiten, Smalo, *Representation theory of Artin algebras* (for Auslander algebras).

- **Session 7: The Strong No Loops Conjecture**.
  - Proof of the no loops conjecture, and the strong no loops conjecture.

  References:
  (1) K. Igusa, S. Liu and C. Paquette, *A proof of the strong no loop conjecture*, ArXiv:1103.5361 (9 pages).
  (2) K. Igusa, *Notes on the no loops conjecture*, J. Pure Appl. Algebra **69** (1990), no. 2, 161–176.

- **Session 8: Finiteness of Representation Dimension**.
  - Relationship of quasi-hereditary algebras to all finite dimensional algebras via endomorphisms of a projective.
  - Statement and proof that any Artin algebra has finite representation dimension, and the relationship to other homological conjectures.

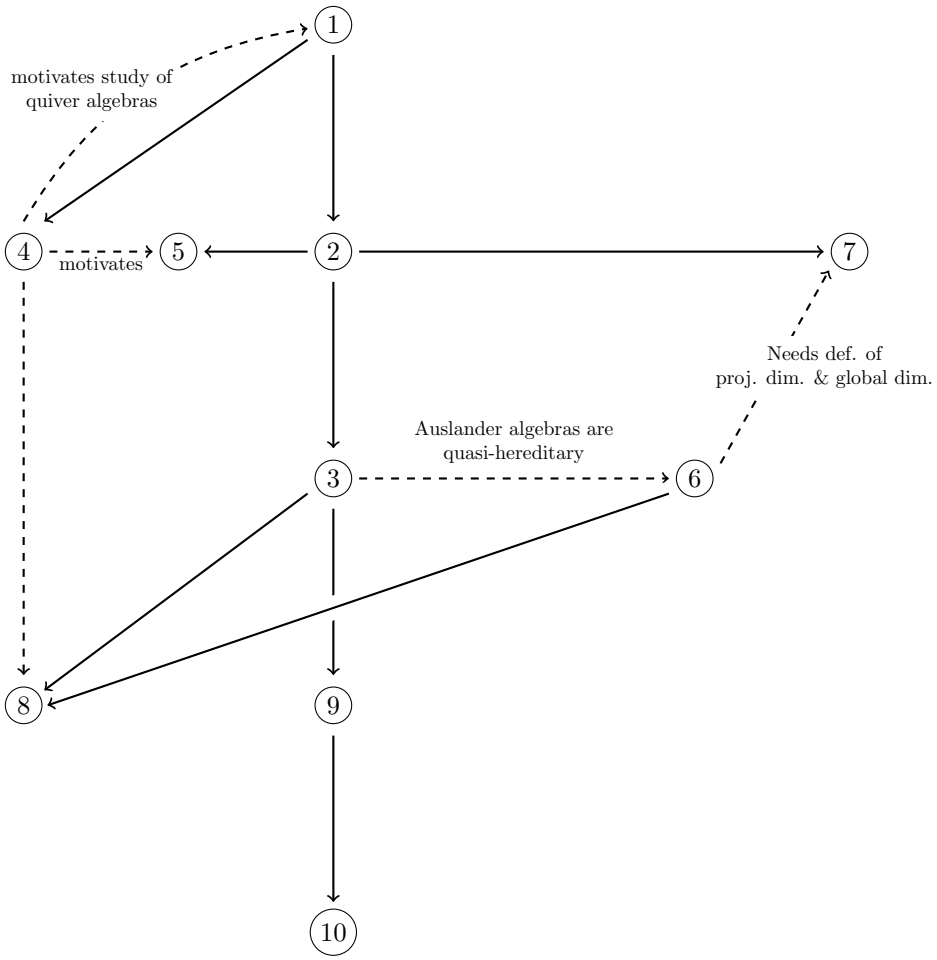
  References:
  (1) O. Iyama, *Finiteness of representation dimension*, Proc. Amer. Math. Soc. **131** (2003), no. 4, 1011–1014.
  (2) C. Ringel, *An introduction to the representation dimension of Artin algebras*. Lecture notes available here.
  (3) C. Ringel, *Iyama's finiteness theorem via strongly quasi-hereditary algebras* available here (11 pages).

- **Session 9: Higher AR theory and examples**.
  - Higher AR translate τ_n, maximal orthogonal subcategories (=cluster tilting).
  - Existence of higher AR sequences.
  - Examples. Preprojective algebras of Dynkin type have a 2-cluster tilting object in $\operatorname{mod}\Lambda$, and GLS relationship to Lie theory.

  References:
  (1) O. Iyama, *Higher dimensional Auslander-Reiten theory on maximal orthogonal subcategories*, ArXiv:0407052.
  (2) Overview in: O. Iyama, *Auslander-Reiten theory revisited*, ArXiv:0803.2841.
  (3) Overview in: O. Iyama, *Cluster tilting for higher Auslander algebras*, ArXiv:0809.4897.

- **Session 10: Higher Auslander Correspondence**.
  - Statement and sketch proof of Iyama's Higher Auslander Correspondence.

  References:
  (1) O. Iyama, *Auslander Correspondence*, ArXiv:0411631.

- **Possible Further Topics**. If there is time, a selection of the following:
  - Riedtmann classification of representation-finite self-injective algebras.
  - Knitting on AR quivers.
  - Recent advances in higher AR theory.

Road map