ON THE DEGENERATION OF THE NONCOMMUTATIVE HODGE-TO-DE RHAM SPECTRAL SEQUENCE

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Abstract. These notes contain an exposition of (parts of) Kaledin's proof of the degeneration of the noncommutative Hodge-to-de Rham spectral sequence for smooth and proper DG-algebras, defined over a field K of characteristic zero. The emphasis is on the analogies with the Deligne-Illusie proof of the degeneration of the Hodge-to-de Rham spectral sequence for smooth and proper schemes over K, which is also discussed in some detail.

1. Introduction

For a variety X defined over the complex numbers $\mathbb{C}$, there is a Hodge spectral sequence

$$E_1^{ab} = H^b(X, \Omega^a_{X/\mathbb{C}}) \Rightarrow H^{a+b}_{dR}(X/\mathbb{C}), \tag{1.1}$$

which degenerates at the first page if X is smooth and proper. The classical proof of this degeneration statement uses GAGA and the Hodge decomposition, and is thus analytic in nature.

For a DG-algebra A over $\mathbb{C}$, the analogue of left hand side of (1.1) is given by the Hochschild homology groups $\mathrm{HH}_i(A)$ and similarly, the analogue of $H^i_{dR}(X/\mathbb{C})$ is the periodic cyclic homology $\mathrm{HP}_i(A)$. Moreover, there is a spectral sequence

$$\mathrm{HH}_*(A)((u)) \Rightarrow \mathrm{HP}_*(A) \tag{1.2}$$

where u is a formal variable of degree 2. This leads to the Kontsevich-Soibelman degeneration conjecture:

Conjecture 1.1. [10, Conjecture 9.1.2] Assume that A is a smooth and proper DG-algebra defined over a field of characteristic zero. Then (1.2) degenerates at the first page.

This conjecture has been settled positively by Kaledin in [8]. The proof is (loosely) based on an algebraic proof of the degeneration of (1.1), due to Deligne and Illusie [2].

Kaledin's proof requires various results from his earlier papers [3, 4, 5, 6, 7, 9]. The reason for this cornucopia is that the general result emerged only gradually: Kaledin first considered the problem for sheaves of associative algebras over a site, then for DG-algebras concentrated in non-negative degrees, and finally for general DG-algebras.

A proof for DG-algebras concentrated in non-positive degrees has also been found by Shklyarov [15], and very recently Akhil Mathew [11] has given a proof of the general case, using the properties of topological Hochschild homology and the reformulation, due to Nikolaus and Scholze [12], of the theory of cyclotomic spectra. Kaledin seems to have been fully aware of the fact that topological Hochschild homology provides a good framework for the conjecture, see the introduction to [4].

In these notes, which are mostly elementary and expository, we first give an overview of the Deligne-Illusie proof, before discussing some aspects of Kaledin's proof of Conjecture 1.1.

Warning 1.2. This is a still a rough version which will (hopefully) get updated with more details soon. Nevertheless, if you spot any typos or mistakes, do let me know.

2. Hodge-to-de Rham following Deligne-Illusie

2.1. **Two spectral sequences associated to the de Rham complex.** Consider abelian categories $\mathcal{A}$ and $\mathcal{B}$, such that $\mathcal{A}$ has enough injectives, and a left exact functor $F : \mathcal{A} \to \mathcal{B}$. Let C^* denote a bounded complex in $\mathcal{A}$. Then there are two spectral sequences, which both compute the hypercohomology $\mathbb{R}F(C^*)$ of C^*:

$$\begin{aligned} E_1^{ab} &= R^bF(C^a) \Rightarrow \mathbb{R}^{a+b}F(C^*) \\ E_2^{ab} &= R^aF(H^b(C^*)) \Rightarrow \mathbb{R}^{a+b}F(C^*), \end{aligned}$$

arising from the stupid, respectively canonical, filtration of C^*. If X denotes a variety over a field k, set

$$\begin{aligned} \mathcal{A} &= Ab(X), \text{ the category of sheaves of abelian groups on } X \\ C^* &= (\Omega^*_{X/k}, d), \text{ the de Rham complex} \\ \mathcal{B} &= Ab, \text{ the category of abelian groups} \\ F &= \Gamma, \text{ the global sections functor} \end{aligned}$$

Then the corresponding spectral sequences look as follows:

$$E_1^{ab} = H^b(X, \Omega^a_{X/k}) \Rightarrow H^{a+b}_{dR}(X/k) \tag{2.1}$$

$$E_2^{ab} = H^a(X, \mathcal{H}^b(\Omega^*_{X/k}) \Rightarrow H^{a+b}_{dR}(X/k) \tag{2.2}$$

The first one (2.1) is called the Hodge spectral sequence (HSS), while (2.2) is called the conjugate spectral sequence (CSS). The (HSS) allows us to formulate the classical Hodge-to-de Rham degeneration theorem.

Theorem 2.1. Let k be a field of characteristic 0 and X/k a smooth proper variety. Then (HSS) degenerates on the first page.

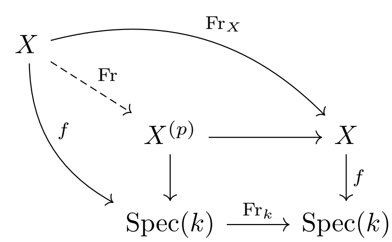
For k of characteristic zero, the sheaves $\mathcal{H}^b(\Omega^*_{X/k}, d)$ appearing in (CSS) are difficult to compute. There are essentially two reasons for this: since the de Rham complex is not a resolution of the constant sheaf $\underline{k}$ (we are in the Zariski topology), the higher cohomology sheaves have no reason to be zero (in contrast to the complex analytic setting). Moreover, since the de Rham differential is not $\mathcal{O}_X$-linear, $\mathcal{H}^b(\Omega^*_{X/k}, d)$ is not necessarily a sheaf of $\mathcal{O}_X$-modules.

2.2. **The Frobenius morphism and Cartier isomorphism.** The situation changes drastically in positive characteristic. To explain this, we need some positive characteristic differential calculus.

Assume X is a k-scheme, where k is a perfect field of characteristic $p > 0$. Denote by $f : X \to \operatorname{Spec}(k)$ the structure morphism. Then the absolute Frobenius morphism

$$\mathrm{Fr}_X : X \to X$$

is defined to be the identity on the underlying topological spaces, and the p-th power map on local sections. This is not a morphism of k-schemes, which can easily be fixed by considering the relative Frobenius morphism $\mathrm{Fr} = \mathrm{Fr}_{X/k}$. This morphism is constructed in the following commuting diagram:



More precisely, one first constructs the Frobenius twist $X^{(p)}$ as the pullback of f and Fr_k, and then uses the universal property of pullbacks to obtain Fr.

Remark 2.2. Since k is perfect, Fr_k is an isomorphism, and hence the morphism $X^{(p)} \to X$ in the diagram is an isomorphism of schemes, but not an isomorphism of k-schemes.

We will use the following basic facts about the Frobenius morphism.

Lemma 2.3. (1) If X is smooth of dimension n, then Fr is finite, flat and the $\mathcal{O}_{X^{(p)}}$-algebra $\mathrm{Fr}_* \mathcal{O}_X$ is locally free of rank p^n.
(2) If X is smooth and projective, then there are isomorphisms

$$H^a(X, \Omega^b_{X/k}) \cong H^a(X^{(p)}, \Omega^b_{X^{(p)}/k}).$$

The proof of the next proposition follows from a local computation using the formula

$$d(s^p t) = s^p d(t)$$

for local sections s, t of $\mathcal{O}_X$.

Proposition 2.4. The differential of the complex $\mathrm{Fr}_* \Omega^*_{X/k}$ is $\mathcal{O}_{X^{(p)}}$-linear.

This proposition ensures one can compare the cohomology sheaves of $\mathrm{Fr}_* \Omega^*_{X/k}$, which are hence $\mathcal{O}_{X^{(p)}}$-modules, with the differential forms on $X^{(p)}$. This leads to the following miracle of positive characteristic differential calculus.

Theorem 2.5 (The Cartier isomorphism)**.** For a smooth k-scheme X, there exists a unique isomorphism of graded $\mathcal{O}_{X^{(p)}}$-algebras

$$\gamma = \bigoplus_{i \geq 0} \gamma^i : \bigoplus_{i \geq 0} \Omega^i_{X^{(p)}/k} \to \bigoplus_{i \geq 0} \mathcal{H}^i \, \mathrm{Fr}_* \Omega^*_{X/k},$$

such that:

(1) $\gamma^0 = \mathrm{Fr}^* : \mathcal{O}_{X^{(p)}/k} \to \mathrm{Fr}_* \mathcal{O}_X$,
(2) $\gamma^1 : \Omega^1_{X^{(p)}/k} \to \mathcal{H}^1 \mathrm{Fr}_* \Omega^*_{X/k} : d(1 \otimes s) \mapsto [s^{p-1} ds]$,

for s a section of $\mathcal{O}_X$, and where the algebra structure on both sides comes from the exterior product.

This isomorphism provides the link between the conjugate and Hodge spectral sequences, which does not seem to exist in characteristic zero.

Corollary 2.6. Let X/k be a smooth proper variety. Then (HSS) degenerates at the first page iff (CSS) degenerates at the second page.

Proof. We calculate:

$$\begin{aligned} H^a(X, \mathcal{H}^b(\Omega^*_{X/k})) &\cong H^a(X^{(p)}, \mathcal{H}^b(\mathrm{Fr}_* \Omega^*_{X/k})) \\ &\cong H^a(X^{(p)}, \Omega^b_{X^{(p)}/k}) \\ &\cong H^a(X, \Omega^b_{X/k}) \end{aligned}$$

so one obtains isomorphisms

$$E_2^{a,b} = H^a(X, \mathcal{H}^b(\Omega^*_{X/k})) \cong H^a(X, \Omega^b_{X/k}) = E_1^{b,a}.$$

Since both spectral sequences also have the same abutment, we are done. □

2.3. **Degeneration in positive characteristic.** We still assume k is a perfect field of characteristic $p > 0$. Abusing notation, we will also denote the differential in the pushforward $\mathrm{Fr}_* \Omega^*_{X/k}$ of the de Rham complex by d. As an immediate consequence of the Cartier isomorphism, we see (and have already used) that the complexes $(\Omega^*_{X^{(p)}/k}, 0)$ and $(\mathrm{Fr}_* \Omega^*_{X/k}, d)$ have isomorphic cohomology. It is hence natural to ask:

Question 2.7. When does the canonical filtration on the de Rham complex split? Or equivalently, does there exist a quasi-isomorphism

$$(\Omega^*_{X^{(p)}/k}, 0) \to (\mathrm{Fr}_* \Omega^*_{X/k}, d) \tag{2.3}$$

inducing the Cartier isomorphism γ in cohomology?

If this question has a positive answer, we will say that (QI) holds. This turns out to be the key to prove degeneration.

Proposition 2.8. If X/k is smooth and proper, and (QI) holds, then (HSS) degenerates at the first page.

Proof. We calculate:

$$\begin{aligned} H^i_{dR}(X/k) &= \mathbb{H}^i(X, \Omega^*_{X/k}) \\ &\cong \mathbb{H}^i(X^{(p)}, \mathrm{Fr}_* \Omega^*_{X/k}) \\ &\cong \mathbb{H}^i(X^{(p)}, \Omega^*_{X^{(p)}/k}) \\ &\cong \oplus_{a+b=i} H^a(X^{(p)}, \Omega^b_{X^{(p)}/k}) \\ &\cong \oplus_{a+b=i} H^a(X, \Omega^b_{X/k}) \end{aligned}$$

□

The main difficulty is hence to check when (QI) holds. Deligne and Illusie give several sufficient conditions, but we will only discuss the following:

Theorem 2.9. Let X/k be a smooth proper variety. Then (QI) holds if X/k has a smooth lift to the ring of Witt vectors $W_2(k)$ of length 2 over k and $\dim(X) < p$.

Remark 2.10. Conditions are needed in positive characteristic: there are smooth and proper surfaces for which (HSS) does not degenerate, see for example [2, §2.6, 2.10].

Let us briefly discuss the proof and the appearance of $W_2(k)$. In order to lift γ, the first non-trivial task is to lift γ^1. There is a natural candidate here, given by the pullback:

$$\mathrm{Fr}^* : \Omega^1_{X^{(p)}/S} \to \mathrm{Fr}_* \, \Omega^1_{X/S},$$

but one can show that this morphism is zero: this comes down to

$$d(s^p) = ps^{p-1}ds = 0. \tag{2.4}$$

The middle term in (2.4) already looks a lot like γ^1, except for the factor of p. Hence, it would be great if we could divide by p in Fr^*.

This is where $W_2(k)$ comes in. The ring of Witt vectors of length 2 is a lift of k as in the following diagram:

$$\begin{array}{ccc} S := \mathrm{Spec}(k) & \longrightarrow & T := \mathrm{Spec}(W_2(k)) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{F}_p) & \longrightarrow & \mathrm{Spec}(\mathbb{Z}/p^2\mathbb{Z}) \end{array}$$

More precisely, it is uniquely characterized by the following two properties:

(1) $W_2(k)$ is flat over $\mathbb{Z}/p^2\mathbb{Z}$,
(2) $W_2(k)/pW_2(k) \cong k$.

Let us assume for now that not only X but also $\mathrm{Fr} : X \to X^{(p)}$ lifts to $W_2(k)$, i.e. there is a diagram:

$$\begin{array}{lccccc} \widetilde{\mathrm{Fr}} : & \widetilde{X} & \longrightarrow & & \longrightarrow & \widetilde{X^{(p)}} \\ & \uparrow & \searrow & T & \swarrow & \uparrow \\ & & & \uparrow & & \\ \mathrm{Fr} : & X & \longrightarrow & & \longrightarrow & X^{(p)} \\ & & \searrow & S & \swarrow & \end{array}$$

Then one shows that there is a factorization

$$\Omega^1_{\widetilde{X^{(p)}}/T} \dashrightarrow p\widetilde{\mathrm{Fr}}_*\Omega^1_{\widetilde{X}/T} \hookrightarrow \widetilde{\mathrm{Fr}}_*\Omega^1_{\widetilde{X}/T}$$

where the composite is $\widetilde{\mathrm{Fr}}^*$.

which in turn induces a factorization

$$\begin{array}{ccc}
\Omega^1_{\widetilde{X^{(p)}}/T} & \xrightarrow{\widetilde{\mathrm{Fr}}^*} & p\widetilde{\mathrm{Fr}}_*\Omega^1_{\widetilde{X}/T} \\
\downarrow & & p\uparrow\cong \\
\Omega^1_{X^{(p)}/S} & \overset{\text{“}\widetilde{\mathrm{Fr}}^*/p\text{”}}{\dashrightarrow} & \mathrm{Fr}_*\,\Omega^1_{X/S}
\end{array}$$

The dashed morphism induces γ^1 in cohomology and can be extended multiplicatively to the desired quasi-isomorphism (2.3), giving a positive answer to (QI). So in a sense, being able to lift to $W_2(k)$ allows one to “divide by p”, which is exactly what was needed to lift the Cartier isomorphism to a quasi-isomorphism.

In general, even if one assumes X lifts globally to $W_2(k)$, the Frobenius only lifts locally, and Deligne and Illusie showed that the corresponding local morphisms $\widetilde{\mathrm{Fr}}^*/p$ we constructed above can be glued to a globally defined morphism

$$\phi^1 : \Omega^1_{X^{(p)}/S}[-1] \to \mathrm{Fr}_*\,\Omega^*_{X/S}$$

in the derived category of coherent sheaves on $X^{(p)}$. By tensoring this morphism one constructs morphisms

$$\phi^i : \Omega^i_{X^{(p)}/S}[-i] \to (\Omega^1_{X^{(p)}/S}[-1])^{\otimes i} \xrightarrow{(\phi^1)^{\otimes i}} (\mathrm{Fr}_*\,\Omega^*_{X/S})^{\otimes i} \to \mathrm{Fr}_*\,\Omega^*_{X/S},$$

where the first morphism is the usual anti-symmetrization map. This anti-symmetrization map requires one to divide by $\dim(X)$, and explains the condition $\dim(X) < p$.

2.4. **From characteristic p to characteristic 0.** Let us now finally prove Theorem 2.1. Assume X is a smooth proper scheme, defined over a field K of characteristic zero. Set $h^{i,j} = \dim_K(H^j(X, \Omega^i_{X/K}))$ and $h^n = \dim_K(H^n_{dR}(X/K))$. It suffices to prove that

$$\sum_{i+j=n} h^{ij} = h^n.$$

Since X is smooth and proper, there exists a finitely generated $\mathbb{Z}$-subalgebra $R \subset K$, and a smooth proper scheme $\mathcal{X}$ over $S = \mathrm{Spec}(R)$, such that

$$\begin{array}{ccc}
X & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow f \\
\mathrm{Spec}(K) & \longrightarrow & S.
\end{array}$$

Up to replacing R by $R[t^{-1}]$, for some $0 \neq t \in R$, we can assume S is smooth over $\mathrm{Spec}(\mathbb{Z})$, and the sheaves $R^j f_*\Omega^i_{\mathcal{X}/S}$ and $R^n f_*\Omega^*_{\mathcal{X}/S}$ are locally free of rank h^{ij}, respectively h^n. Choose an integer d which bounds the dimensions of the fibers of $\mathcal{X}$ over S at any point of S. One can now choose a point $s \in S$, for which the (finite!) residue field $k = k(s)$ is of characteristic $p > d$ (because S has “enough” closed points). One obtains a commutative diagram

$$\begin{array}{ccccccc}
Y & \longrightarrow & Y_1 & \longrightarrow & \mathcal{X} & \longleftarrow & X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
s = \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(W_2(k)) & \xrightarrow{g} & S & \longleftarrow & \mathrm{Spec}(K)
\end{array}$$

where g is the extension of the closed immersion $\mathrm{Spec}(k) \to S$, which exists since S is smooth over $\mathrm{Spec}(\mathbb{Z})$, Y is the fibre of $\mathcal{X}$ over s, and Y_1 is induced from $\mathcal{X}$ from the base change g. We can now apply Theorem 2.9 to Y to obtain

$$\sum_{i+j=n} \dim_k(H^j(Y, \Omega^i_{Y/k})) = \dim_k(H^n_{dR}(Y/k)).$$

But by the discussion on $R^j f_* \Omega^i_{\mathcal{X}/S}$ and $R^n f_* \Omega^*_{\mathcal{X}/S}$ above, we have

$$h^{ij} = \dim_k(H^j(Y, \Omega^i_{Y/k})),$$
$$h^n = \dim_k(H^n_{dR}(Y/k)),$$

and hence we are done.

3. Hochschild-to-periodic cyclic following Kaledin

Let A denote a DG-algebra defined over a field k. Then as we will see, there is a spectral sequence

$$\mathrm{HH}_*(A)((u)) \Rightarrow \mathrm{HP}_*(A) \tag{3.1}$$

where u is a formal generator of degree 2. This is the noncommutative Hodge spectral sequence (NC-HSS), and the degeneration conjecture (now a theorem) says:

Conjecture 3.1 (Kontsevich-Soibelman)**.** Assume that A is a smooth and proper DG-algebra defined over a field of characteristic zero. Then (NC-HSS) degenerates.

3.1. **Proof strategy.** The main steps in Kaledin's proof of Conjecture 3.1 are as follows:

(1) For k of positive characteristic, construct a noncommutative analogue (NC-CSS) of the conjugate spectral sequence (CSS).
(2) For smooth and proper DG-algebras, identify the first pages and the abutments of (NC-CSS) and (NC-HSS).
(3) Identify necessary conditions ensuring the degeneration of (NC-CSS).
(4) For a smooth and proper DG-algebra defined over a field of characteristic zero, deduce the degeneration of (NC-HSS) by reduction to positive characteristic.

We will briefly discuss each of these steps, but for expository reasons, we will usually restrict to the baby case, where A is an associative k-algebra, and k is an algebraically closed field. As we will see, Conjecture 3.1 is then easily established, but the techniques used by Kaledin to settle the general case give interesting and non-trivial statements even for associative algebras.

3.2. **Smooth and proper DG-algebras.** In this section, we briefly define and discuss the DG-algebras appearing in the statement of Conjecture 3.1. Let k denote a base commutative ring and A a DG-algebra over k, such that A^i is flat over k.

Denote by $D(A)$ the derived category of DG A-modules, which is obtained from the abelian category of DG A-modules by formally inverting the quasi-isomorphisms: by definition these are the morphisms of DG A-modules that are quasi-isomorphisms of the underlying complexes of k-modules. The derived category $D(A)$ is triangulated, so we can consider the category of perfect complexes $\mathrm{D}^{\mathrm{perf}}(A)$, i.e. the smallest thick triangulated subcategory of $D(A)$ containing A.

Definition 3.2. A DG-algebra A over k is called:

(1) proper: if $A \in \mathrm{D}^{\mathrm{perf}}(k)$,
(2) smooth: if $A \in \mathrm{D}^{\mathrm{perf}}(A^e)$,

where $A^e := A^{\mathrm{op}} \otimes_k A$ denotes the enveloping DG-algebra of A.

Example 3.3. Let k denote an algebraically closed field, and A an associative k-algebra (considered as DG-algebra concentrated in degree zero). Then A is smooth and proper iff A is a finite dimensional algebra of finite global dimension.

Example 3.4. Let A be a commutative k-algebra essentially of finite type.[1] Then A is smooth iff $\mathrm{Spec}(A)$ is smooth.

One of the main justifications for Definition 3.2 comes from geometry. Let X be a scheme over a field k. A complex of sheaves of $\mathcal{O}_X$-modules is perfect if it is locally quasi-isomorphic to a bounded complex of vector bundles. Denote by $D(X)$ the full subcategory of the derived category of sheaves of $\mathcal{O}_X$-modules consisting of complexes with quasi-coherent cohomology and by $\mathrm{D}^{\mathrm{perf}}(X)$ the full subcategory of $D(X)$ consisting of perfect complexes.

Theorem 3.5. [1, Corollary 3.1.8] Assume that X is a quasi-compact and quasi-separated scheme over a field k. Then $D(X)$ is equivalent to $D(A)$ for a DG-algebra A with bounded cohomology.

Theorem 3.6. [14, Proposition 3.31] Let X be a separated scheme of finite type over a field k. Then X is smooth and proper if and only if A is smooth and proper.

Implicit in the statement of these theorems is that smoothness and properness of A only depend on $D(A)$.

Example 3.7. If X is a smooth projective variety, and $\mathcal{L}$ denotes a very ample line bundle, then by [13, Theorem 4], the DG-algebra A can be chosen as follows:

$$A = \mathrm{RHom}(\oplus_{i=0}^{\dim X} \mathcal{L}^{\otimes i}, \oplus_{i=0}^{\dim X} \mathcal{L}^{\otimes i}).$$

The cohomology of A is clearly concentrated in non-negative degrees, but the stronger statement that A itself is concentrated in non-negative degrees is also true.

3.3. **The noncommutative conjugate spectral sequence.** Let's recall the construction of the (NC-HSS) for an associative algebra A. There is a periodic cyclic

[1] This means that A is a localization of a commutative k-algebra of finite type.

bicomplex

$$
\begin{array}{ccccccccc}
\mathrm{CC}^{\mathrm{per}}: & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \\
\cdots & \xleftarrow[1-t]{} & A^{\otimes 3} & \xleftarrow[N]{} & A^{\otimes 3} & \xleftarrow[1-t]{} & A^{\otimes 3} & \xleftarrow[N]{} & \cdots \\
 & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \\
\cdots & \xleftarrow[1-t]{} & A^{\otimes 2} & \xleftarrow[N]{} & A^{\otimes 2} & \xleftarrow[1-t]{} & A^{\otimes 2} & \xleftarrow[N]{} & \cdots \\
 & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \\
\cdots & \xleftarrow[1-t]{} & A & \xleftarrow[N]{} & A & \xleftarrow[1-t]{} & A & \xleftarrow[N]{} & \cdots \\
 & & & & & & & & \\
\text{column } \#: & & -1 & & 0 & & 1 & &
\end{array}
$$

where b' is the bar differential, b is the Hochschild differential,

$$t(a_1 \otimes \cdots \otimes a_k) = (-1)^{k+1} a_2 \otimes \cdots \otimes a_k \otimes a_1,$$

and

$$N := 1 + t + \cdots + t^{k-1} : A^{\otimes k} \to A^{\otimes k}.$$

The periodic cyclic homology of A is then defined as

$$\mathrm{HP}_*(A) := H_*(\mathrm{Tot}^{\Pi}(\mathrm{CC}^{\mathrm{per}})). \tag{3.2}$$

Since the odd degree columns in the bicomplex are acyclic, we hence obtain the (NC-HSS)

$$\mathrm{HH}_*(A)((u)) \Rightarrow \mathrm{HP}_*(A),$$

where u is a formal variable of degree 2.

Let's check Conjecture 3.1 is true in this case. As we saw in Example 3.3, an associative algebra A is smooth and proper iff A is finite dimensional of finite global dimension. It is then well known[2] that

$$\mathrm{HH}_i(A) = \begin{cases} k^{|A|} & i = 0 \\ 0 & i \neq 0 \end{cases} \qquad \mathrm{HP}_i(A) = \begin{cases} k^{|A|} & i = 2n \\ 0 & i = 2n+1 \end{cases}$$

where $|A|$ denotes the number of isomorphism classes of simple A-modules. This clearly implies that (3.1) degenerates.

In order to imitate the Deligne-Illusie proof, it seems we need an analogue of the Frobenius morphism. Of course, for an associative, but not necessarily commutative, algebra A defined over a field of characteristic $p > 0$, the map

$$\mathrm{fr} : A \to A : a \mapsto a^p$$

[2] This also follows because the noncommutative motive of A only depends on the field k, see (3.7).

is not a ring morphism: it is neither additive nor multiplicative. Factoring fr as follows:

$$\begin{array}{ccccc} & & \overset{\text{fr}}{\frown} & & \\ A & \xrightarrow[\phi]{} & A^{\otimes p} & \xrightarrow[m]{} & A \\ a & \longrightarrow & a^{\otimes p} & \longrightarrow & a^p \end{array}$$

we will see that, in a loose sense, step (1) of Kaledin's proof is concerned with m, whereas step (3) focusses on ϕ.

3.4. **Step (1).** In the definition (3.2) of periodic cyclic homology, we used the product total complex. This is explained by the following lemma.

Lemma 3.8. If k is of characteristic zero, then

$$H_*(\mathrm{Tot}^{\bigoplus}(\mathrm{CC}^{\mathrm{per}})) = 0$$

Now assume k is a perfect field of odd characteristic $p > 0$. In this case, Lemma 3.8 is no longer correct, and the "wrong" totalization rise to a new invariant, the co-periodic cyclic homology:

$$\overline{\mathrm{HP}}_*(A) := H_*(\mathrm{Tot}^{\bigoplus}(\mathrm{CC}^{\mathrm{per}})).$$

Co-periodic cyclic homology is in fact defined for any DG-algebra over a commutative ring k, and Kaledin shows that it is a derived Morita-invariant, and gives an additive invariant of small DG-categories, as soon as k is noetherian.

Denote by $A^{(p)} := A \otimes_k k$, where the k-structure on k comes from the Frobenius morphism $\mathrm{Fr}_k : k \to k$. Inspired by the Cartier isomorphism, one might expect an analogue of the (CSS) to provide a relation between $\mathrm{HH}_*(A^{(p)})$ and $\mathrm{HP}_*(A)$. This is not quite correct: as it turns out, the correct analogue of (CSS) is provided by a spectral sequence

$$\mathrm{HH}_*(A^{(p)})((u^{-1})) \Rightarrow \overline{\mathrm{HP}}_*(A) \tag{3.3}$$

converging to the co-periodic cyclic homology of A, where u is again a formal variable of degree 2. This is the noncommutative conjugate spectral sequence (NC-CSS).

In contrast to the commutative case, no analogue of the conjugate spectral sequence is known in characteristic zero, and in the generality stated, its construction is very subtle.

Since the periodic bicomplex is an upper half plane bicomplex (and is hence zero in the fourth quadrant), it gives rise to a spectral sequence

$$\check{\mathrm{H}}_j(\mathbb{Z}/(i+1)\mathbb{Z}, A^{\otimes i+1}) \Rightarrow \overline{\mathrm{HP}}_{i+j}(A), \tag{3.4}$$

where $\check{\mathrm{H}}_*(\mathbb{Z}/(i+1)\mathbb{Z}, -)$ denotes the Tate homology of the cyclic group $\mathbb{Z}/(i+1)\mathbb{Z}$, which acts on $A^{\otimes i+1}$ by permuting the factors. If p does not divide $i+1$, then $\check{\mathrm{H}}_*(\mathbb{Z}/(i+1)\mathbb{Z}, A^{\otimes i+1}) = 0$, since the group algebra $k\mathbb{Z}/(i+1)\mathbb{Z}$ is semisimple, but if $i + 1 = lp$ for some $l \in \mathbb{Z}$ this is no longer the case. To see what happens then, Kaledin considers a "p-fattened" version of the periodic bicomplex:

$$
\mathrm{CC}^{\mathrm{per}}_{\mathrm{p}}:\quad
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle -b'_p} & & \downarrow{\scriptstyle b_p} & & \downarrow{\scriptstyle -b'_p} & & \\
\cdots & \xleftarrow[1-t]{} & A^{\otimes 3p} & \xleftarrow[N]{} & A^{\otimes 3p} & \xleftarrow[1-t]{} & A^{\otimes 3p} & \xleftarrow[N]{} & \cdots \\
 & & \downarrow{\scriptstyle -b'_p} & & \downarrow{\scriptstyle b_p} & & \downarrow{\scriptstyle -b'_p} & & \\
\cdots & \xleftarrow[1-t]{} & A^{\otimes 2p} & \xleftarrow[N]{} & A^{\otimes 2p} & \xleftarrow[1-t]{} & A^{\otimes 2p} & \xleftarrow[N]{} & \cdots \\
 & & \downarrow{\scriptstyle -b'_p} & & \downarrow{\scriptstyle b_p} & & \downarrow{\scriptstyle -b'_p} & & \\
\cdots & \xleftarrow[1-t]{} & A^{\otimes p} & \xleftarrow[N]{} & A^{\otimes p} & \xleftarrow[1-t]{} & A^{\otimes p} & \xleftarrow[N]{} & \cdots \\
 & & & & & & & & \\
\text{column } \#: & & -1 & & 0 & & 1 & &
\end{array}
\tag{3.5}
$$

where $t : A^{\otimes lp} \to A^{\otimes lp}$ is again the cyclic permutation (now of order lp) twisted by a sign,

$$b'_p = \sum_{0\le i\le l} (-1)^i \mathrm{id}^{\otimes pi} \otimes m^{\otimes p} \otimes \mathrm{id}^{\otimes p(l-i)} : A^{\otimes p(l+2)} \to A^{\otimes p(l+1)},$$

where $m : A \otimes A \to A$ denotes the multiplication,

$$b_p = b'_p + (-1)^{p(l+2)}(m \otimes \mathrm{id}^{p(l-2)-2}) \circ t : A^{\otimes p(l+2)} \to A^{\otimes p(l+1)}$$

and

$$N = 1 + t + \ldots + t^{lp-1} : A^{\otimes lp} \to A^{\otimes lp}.$$

Remember that $\mathrm{HH}_*(A)$ can be computed from the homology of $\mathrm{Bar}_*(A) \otimes_{A^e} A$, where $\mathrm{Bar}_*(A)$ denotes the bar resolution. Of course, we can use any other free A-bimodule resolution of A, and in particular we can compute $\mathrm{HH}_*(A)$ from the homology of

$$(\mathrm{Bar}_*(A)^{\otimes_A p}) \otimes_{A^e} A,$$

which is exactly the complex appearing in the even columns of (3.5). In fact, it turns out that the bicomplex (3.5) can also be used as an alternative way to compute cyclic and (co-)periodic cyclic homology.

Proposition 3.9. There are isomorphisms:

1. $H_*(\mathrm{Tot}^{\Pi}(\mathrm{CC}^{\mathrm{per}}_{\mathrm{p}})) \cong \mathrm{HP}_*(A)$
2. $H_*(\mathrm{Tot}^{\bigoplus}(\mathrm{CC}^{\mathrm{per}}_{\mathrm{p}})) \cong \overline{\mathrm{HP}}_*(A)$

This method (i.e. passing from $\mathrm{CC}^{\mathrm{per}}$ to $\mathrm{CC}^{\mathrm{per}}_{\mathrm{p}}$) is known as edgewise subdivision. Proposition 3.9 is quite subtle to prove, since there is no map between the bicomplexes $\mathrm{CC}^{\mathrm{per}}$ and $\mathrm{CC}^{\mathrm{per}}_{\mathrm{p}}$ inducing these isomorphisms. In order to keep track of the combinatorics, and prove statements like Proposition 3.9, it is better to use homology of small categories, and in particular the cyclic category and its p-fold cover, which we do not discuss here. The (second page) analogue of (3.4) for $\mathrm{CC}^{\mathrm{per}}_{\mathrm{p}}$ now looks like

$$H^v_i(\check{\mathrm{H}}_j(\mathbb{Z}/p(i+1)\mathbb{Z}, A^{\otimes p(i+1)})) \Rightarrow \overline{\mathrm{HP}}_{i+j}(A), \tag{3.6}$$

where $H_v^*(-)$ denotes vertical homology. By a careful analysis of the initial term of this spectral sequence, Kaledin shows that this is exactly the spectral sequence (3.3). One of the main steps in this identification is a version of the following lemma for complexes of vector spaces.

Lemma 3.10. For any vector space V over k, the map $\phi : V^{(1)} \to V^{\otimes p} : v \mapsto v^{\otimes p}$ induces isomorphisms:

$$\check{\mathrm{H}}_*(\mathbb{Z}/p\mathbb{Z}, V^{(1)}) \xrightarrow{\cong} \check{\mathrm{H}}_*(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p}),$$

where $V^{(1)}$ carries the trivial $\mathbb{Z}/p\mathbb{Z}$ action.

Since moreover $\check{\mathrm{H}}_*(\mathbb{Z}/p\mathbb{Z}, V^{(1)}) \cong V^{(1)}$, one can use Lemma 3.10 to transform (3.6) into the noncommutative conjugate spectral sequence (3.3).

3.5. **Step (2).** The following theorem shows that in positive characteristic, the first interesting pages of (NC-HSS) and (NC-CSS) agree, just like in the commutative case.

Theorem 3.11. Assume that A is a smooth and proper DG-algebra over a field of finite characteristic. Then

$$\mathrm{HP}_*(A) \cong \overline{\mathrm{HP}}_*(A).$$

In the baby case, for A a finite dimensional of finite global dimension over an algebraically closed field k of positive characteristic, we know by [16, Corollary 3.14] that

$$\mathrm{M}_{\mathrm{nc}}(A) \cong \mathrm{M}_{\mathrm{nc}}(k)^{\oplus s}, \tag{3.7}$$

where s denotes the number of isomorphism classes of simple A-modules, and $\mathrm{M}_{\mathrm{nc}}(-)$ is the noncommutative motive (i.e. the universal additive invariant). We already mentioned that $\overline{\mathrm{HP}}_*$ is an additive invariant, so to compute $\overline{\mathrm{HP}}_*(A)$, it suffices to compute $\overline{\mathrm{HP}}_*(k)$. The periodic cyclic bicomplex is now particularly easy to write down, and one checks by hand that the only non-zero terms on the $(p+1)$-st page of the spectral sequence are on the p-th row, where they alternate between k and 0, confirming Theorem 3.11 in this simple case.

3.6. **Step (3).** Fix a perfect field k of positive characteristic p, and assume A is a DG-algebra over k. To formulate Kaledin's analogue of Theorem 2.9, we first need to introduce some cohomology groups associated to A.

Just like Hochschild cohomology for associative algebras, there is a cohomology theory for DG-algebras, which can be used to describe their first order deformations, but now only up to quasi-isomorphism (in contrast to isomorphism for associative algebras). Somewhat confusingly, the corresponding theory is called reduced Hochschild cohomology. If I denotes the kernel of the multiplication map $A^{\mathrm{op}} \otimes_k A \to A$, then the reduced Hochschild cohomology of A is defined as

$$\overline{\mathrm{HH}}^{i+1}(A) = \mathrm{Ext}^i_{A^e}(I, A)$$

in the category $D(A^e)$. For associative algebras, one can check that this agrees with the usual Hochschild cohomology groups: for $i \geq 1$,

$$\mathrm{HH}^{i+1}(A) = \overline{\mathrm{HH}}^{i+1}(A),$$

which explains why this theory is usually not mentioned in the classical setting. The group $\overline{\mathrm{HH}}^2(A)$ classifies the square-zero extensions of A up to quasi-isomorphism. More precisely, if $\overline{\mathrm{HH}}^2(A) = 0$, then for every square-zero extension $p : A' \to A$ of A by A, there exists a DG-algebra A'' and a map $s : A'' \to A'$ such that the composite $A'' \xrightarrow{s} A' \xrightarrow{p} A$ is a quasi-isomorphism. We will refer to such a map s as a DG-splitting of p.

Remark 3.12. Reduced Hochschild cohomology groups are not derived Morita-invariant. This can be fixed by considering DG-categories instead.

We can now formulate the analogue of Theorem 2.9.

Theorem 3.13. Assume that a DG-algebra A, defined over perfect field k of characteristic p, satisfies the following two conditions:

(1) There exist a DG-algebra $\tilde{A}$ over the ring of Witt vectors $W_2(k)$ of length 2 over k and a quasi-isomorphism $\tilde{A} \otimes^{\mathbb{L}}_{W_2(k)} k \cong A$.
(2) The reduced Hochschild cohomology $\overline{\mathrm{HH}}^i(A)$ vanishes for $i \geq 2p$.

Then the noncommutative conjugate spectral sequence (3.3) for A degenerates.

To provide some motivation for why this theorem is true, consider again an associative algebra A, which we will assume to be smooth and proper. The key to degeneration in the commutative setting was an analysis of the behaviour of the Frobenius morphism under the lift to $W_2(k)$. As we already mentioned, in the noncommutative setting we don't have a Frobenius morphism.

Definition 3.14. A quasi-Frobenius map for A is a $\mathbb{Z}/p\mathbb{Z}$-equivariant algebra map $F : A^{(p)} \to A^{\otimes p}$, which induces the isomorphism

$$\check{\mathrm{H}}_*(\mathbb{Z}/p\mathbb{Z}, A^{(p)}) \xrightarrow{\cong} \check{\mathrm{H}}_*(\mathbb{Z}/p\mathbb{Z}, A^{\otimes p})$$

from Lemma 3.10.

Proposition 3.15. If A admits a quasi-Frobenius morphism, then (NC-CSS) degenerates.

Proof. By tensoring the quasi-Frobenius, we obtain maps

$$F^{\otimes n} : (A^{(p)})^{\otimes n} \to A^{\otimes pn}$$

for all n, and these can be assembled into a map of "p-cyclic objects". This in turn induces a map

$$F' : \mathrm{HH}_*(A^{(p)})((u)) \to \mathrm{HP}_*(A),$$

and it suffices to show that F' is an isomorphism. Since $\check{\mathrm{H}}_*(\mathbb{Z}/p\mathbb{Z}, A^{(p)}) \cong A^{(p)}$, a quasi-Frobenius map is injective, and one concludes that the same is true for F'. The cokernel of F must be a free $k[\mathbb{Z}/p\mathbb{Z}]$-module, which one can use to show that F' is surjective. □

There are however very few algebras admitting such a quasi-Frobenius morphism, and the conditions (1) and (2) in Theorem 3.13 are required to fix this.

TO BE ADDED

3.7. **Step (4).** To finish the degeneration argument, we need to be able to pass from characteristic p to characteristic 0. To do this in the DG setting, we need the following theorem of Toën.

Theorem 3.16. [17] For a smooth and proper DG-algebra A, defined over a field K of characteristic zero, there exists a finitely generated subring $R \subset K$ and a smooth and proper DG-algebra A^R over R such that

$$A \cong A^R \otimes^{\mathbb{L}}_R K.$$

This finally allows us to prove Kaledin's degeneration theorem.

Theorem 3.17. Assume that A is a smooth and proper DG-algebra over a field K of characteristic zero. Then the NC Hodge-to-de Rham spectral sequence (3.1) degenerates, so there is an isomorphism

$$\mathrm{HP}_*(A) \cong \mathrm{HH}_*(A)((u)).$$

Proof. Applying Theorem 3.16 to A, we obtain a smooth and proper DG-algebra A^R over R. Using smoothness and properness, there exists a constant N such that $\mathrm{HH}_i(A^R)$ and $\overline{\mathrm{HH}}^i(A^R)$ are finitely generated over R, and $\mathrm{HH}_i(A^R) = \overline{\mathrm{HH}}^i(A^R) = 0$ for $i \geq N$. By localizing R if necessary, we can assume furthermore that the $\mathrm{HH}_*(A^R)$ are projective over R, that for any maximal ideal $\mathfrak{m} \subset R$, $0 \neq p = \mathrm{char}(R/\mathfrak{m}) \in \mathfrak{m}/\mathfrak{m}^2$, and that $2p > N$.

So for any maximal ideal $\mathfrak{m} \subset R$, we can apply Theorem 3.13 to $A^k = A^R \otimes^{\mathbb{L}}_R k$ for $R/\mathfrak{m} \cong k$, and hence

$$\begin{aligned} \overline{\mathrm{HP}}_*(A^k) &\cong \mathrm{HH}_*(A^k)^{(p)}((u^{-1}) \\ &\cong \mathrm{HH}_*(A^k)^{(p)}((u)) \end{aligned}$$

where we replaced Laurent series in u^{-1} with Laurent series in u, since $\mathrm{HH}_*(A^k)$ is concentrated in a finite range of degrees. Since A^k is also smooth and proper, we can use Theorem 3.11 to obtain an isomorphism

$$\mathrm{HP}(A^k) \cong \mathrm{HH}_*^{(p)}(A^k)((u))$$

of graded k-vector space, so the (NC-HSS) degenerates for A^k.

Finally, since all the $\mathrm{HH}_*(A^R)$ are finitely generated and projective, and the differentials in the (NC-HSS) vanish mod $\mathfrak{m}$, for any maximal ideal, they vanish identically, so (NC-HSS) degenerates for A^R, and hence also for A. □

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