Noncommutative Hodge Theory
Lecture 6: Invariants of noncommutative projective schemes

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Abstract

We sketch Tabuada's calculation of the Hodge-theoretic invariants of noncommutative projective schemes defined by Koszul algebras.

Contents

1 Noncommutative projective schemes

In this lecture we look at some examples where the invariants we have been discussing can be explicitly computed. These are given by noncommutative analogues of simple projective varieties such as projective spaces. Throughout the notes, $\mathbb{K}$ is a field, and all $\mathbb{K}$-algebras are assumed to be Noetherian.

Definition 1.1. A ***(connected) graded ring*** is an associative $\mathbb{K}$-algebra $\mathcal{A}$ whose underlying $\mathbb{K}$-vector space is equipped with a direct sum decomposition $\mathcal{A} = \bigoplus_{n=0}^{\infty} \mathcal{A}_n$ with the following properties:

1. $\mathcal{A}_0 = \mathbb{K}$
2. $\dim \mathcal{A}_n < \infty$ for all $n \geq 0$
3. $\mathcal{A}_n \mathcal{A}_m \subset \mathcal{A}_{n+m}$ for all $m, n \geq 0$

Given a graded ring as above, we, may introduce several categories of graded modules:

- We denote by $\mathsf{gr}(\mathcal{A})$ the category whose objects are finitely generated $\mathbb{Z}$-graded $\mathcal{A}$-modules, and whose morphisms are module homomorphisms that preserve the degrees
- We denote by $\mathsf{tors}(\mathcal{A}) \subset \mathsf{gr}(\mathcal{A})$ the full subcategory consisting of modules that are finite-dimensional as $\mathbb{K}$-vector spaces.
- We denote by $\mathsf{qgr}(\mathcal{A}) = \mathsf{gr}(\mathcal{A})/\mathsf{tors}(\mathcal{A})$ the Verdier quotient.

When $\mathcal{A}$ is commutative, these categories have natural geometric interpretations. The following example is a special case of a general result of Serre, which shows that when $\mathcal{A}$ is commutative and generated by $\mathcal{A}_1$, the category $\mathsf{qgr}(\mathcal{A})$ is equivalent to the category of coherent sheaves on the projective scheme associated to $\mathcal{A}$ by the Proj construction.

Example 1.2. Consider the case in which $\mathcal{A} = \mathbb{K}[x_1, \ldots, x_d]$ is a polynomial ring, equipped with the obvious grading for which the variables $x_1, \ldots, x_d$ have degree one. Thus $\mathsf{Spec}(\mathcal{A}) = \mathbb{A}^n$ is an affine space of dimension d. We have a natural action of the multiplicative group $\mathbb{G}_{\mathrm{m}} = \mathbb{K}^\times$ on $\mathbb{A}^d$ by rescaling, and the grading on $\mathcal{A}$ is precisely the decomposition of $\mathcal{A}$ into weight spaces for this action. Note that the $\mathbb{G}_{\mathrm{m}}$-action on $\mathbb{A}^d \setminus \{0\}$ is free and the quotient is the projective space $\mathbb{P}^{d-1}$.

A grading on an $\mathcal{A}$-module corresponds to a $\mathbb{G}_{\mathrm{m}}$-action on the corresponding coherent sheaf on $\mathbb{A}^d$. Thus $\mathsf{gr}(\mathcal{A})$ is the category of $\mathbb{G}_{\mathrm{m}}$-equivariant coherent sheaves on $\mathbb{A}^d$, and $\mathsf{tors}(\mathcal{A})$ is the subcategory consisting of equivariant sheaves supported at the origin.

The quotient category $\mathsf{qgr}(\mathcal{A}) = \mathsf{gr}(\mathcal{A})/\mathsf{tors}(\mathcal{A})$ is the category of equivariant coherent sheaves on $\mathbb{A}^d \setminus \{0\}$, and the quotient functor $\mathsf{gr}(\mathcal{A}) \to \mathsf{qgr}(\mathcal{A})$ is corresponds to the restriction of sheaves along the inclusion $\mathbb{A}^d \setminus \{0\} \subset \mathbb{A}^d$. Thus an equivariant sheaf on $\mathbb{A}^d$ is the same thing as an ordinary coherent sheaf on $\mathbb{P}^{d-1}$, so that we have an identification $\mathsf{qgr}(\mathcal{A}) \cong \mathsf{Coh}(\mathbb{P}^{d-1})$. □

In this lecture we will consider noncommutative variants of this example, such as the following:

Example 1.3. Suppose that $q_{ij} \in \mathbb{K}$ for $1 \leq i, j \leq n$ satisfy, $q_{ji} = q_{ij}^{-1}$. We define a graded algebra as the quotient

$$\mathcal{A} = \frac{\mathbb{K}\langle x_1, \ldots, x_n \rangle}{(x_i x_j - q_{ij} x_j x_i)_{i<j}}$$

This algebra has a basis given by monomials of the form $x_1^{k_1} \cdots x_n^{k_n}$ so it is isomorphic to $\mathbb{K}[x_1, \ldots, x_d]$ as a $\mathbb{K}$-vector space. We therefore imagine that $\mathsf{qgr}(\mathcal{A})$ is the category $\mathsf{Coh}(\mathbb{P}_q^{d-1})$, where $\mathbb{P}_q^{d-1}$ is a noncommutative analogues of projective space. □

Example 1.4. More generally, there is a notion of a "noncommutative projective space", which is determined by a graded algebra $\mathcal{A}$ that behaves sufficiently like a polynomial ring. In particular, we require that $\mathcal{A}$ has finite global dimension, that the Hilbert series

$$h_{\mathcal{A}}(t) := \sum_{k \geq 0} (\dim \mathcal{A}_k) t^k$$

is the same as that of the polynomial ring, and that a certain homological regularity condition defined by Artin–Schelter [2] is satisfied. (This regularity condition is a noncommutative replacement for the fact that the affine space $\mathbb{A}^d$ is smooth, and it holds for the algebras in Example 1.3.)

The graded algebras corresponding to noncommutative analogues of $\mathbb{P}^2$ were classified by Artin–Tate–Van den Bergh in [1]. The classification of noncommutative projective spaces of dimension ≥ 3 is a major open problem, but many examples are known, and the families that arise as deformations of a commutative $\mathbb{P}^3$ were classified in [3] via quantization of Poisson brackets. □

Example 1.5. In [4], Rogalski and Sierra studied algebras of the form

$$\mathcal{A} = \frac{\mathbb{K}\langle x_1, \ldots, x_4 \rangle}{(x_1(ax_1 - x_3) + x_3(x_1 - ax_3), \ldots, x_1(bx_3 - x_4) + x_4(x_3 - bx_4))}$$

These algebras have many features in common with the polynomial ring in four variables; in particular they have global dimension four and Hilbert series $(1-t)^{-4}$. However, Artin–Schelter's regularity condition fails in this case, so they do not define noncommutative analogues of $\mathbb{P}^3$. In fact, it turns out that the category $\mathsf{qgr}(\mathcal{A})$ is not Ext-finite, so it does not behave like the category of sheaves on any smooth variety. □

2 Determination of the cohomology

We would like to determine the Hodge-theoretic invariants of simple noncommutative projective schemes, like the examples above.

Recall from Lecture 4 that if X is a smooth variety, then the periodic cyclic homology of the category $\mathsf{Coh}(\mathsf{X})$ is the de Rham cohomology of X, and the NC Hodge filtration is the filtration by the columns of the Hodge diamond. For a projective space $\mathsf{X} = \mathbb{P}^{d-1}$, the Hodge diamond is very simple:

$$\mathsf{H}^{p,q}(\mathbb{P}^{d-1}) \cong \begin{cases} 0 & p \neq q \\ \mathbb{K} & p = q \end{cases}$$

Thus we have $\mathsf{HP}_{\mathsf{ev}}(\mathsf{Coh}(\mathbb{P}^{d-1})) \cong \mathbb{K}^{\oplus d}$ and $\mathsf{HP}_{\mathsf{odd}}(\mathsf{Coh}(\mathbb{P}^{d-1})) = 0$. Note that $\mathsf{HP}_\bullet(\mathbb{K}) = \mathbb{K}$ in degree zero. We conclude that when $\mathcal{A} = \mathbb{K}[x_1, \ldots, x_d]$, we have

$$\mathsf{HP}_\bullet(\mathsf{qgr}(\mathcal{A})) \cong \mathsf{HP}_\bullet(\mathbb{K})^{\oplus d}.$$

Given that many of the examples in Section 1 correspond to deformations of $\mathbb{P}^{n-1}$, we may expect them to have the same topological invariants, such as the periodic cyclic homology. In fact, all of the Hodge-theoretic invariants we have looked at so far are also the same, as is demonstrated by the following special case of a recent result of Tabuada [6]:

Theorem 2.1. *Let $\mathcal{A}$ be a connected graded Noetherian $\mathbb{K}$-algebra as above. Suppose further that $\mathcal{A}$ is Koszul of global dimension d, and that the Hilbert series of $\mathcal{A}$ has the form*

$$h_{\mathcal{A}}(t) = \frac{1}{1 - \beta_1 t + \cdots \pm \beta_{d-1} t^{d-1} \mp t^d}$$

Then we have isomorphisms

$$\mathsf{HH}_\bullet(\mathsf{qgr}(\mathcal{A})) \cong \mathsf{HC}_\bullet(\mathsf{qgr}(\mathcal{A})) \cong \mathsf{HP}_\bullet(\mathsf{qgr}(\mathcal{A})) \cong \mathbb{K}^{\oplus d}.$$

where $\mathbb{K}^{\oplus d}$ sits in degree zero. More generally, if $\mathcal{T}$ is a $\mathbb{K}$-linear triangulated category and $E : \mathsf{dgCat}(\mathbb{K}) \to \mathcal{T}$ is a functor that is Morita invariant, localizing and cocontinuous (i.e. preserves sequential colimits), then $E(\mathsf{qgr}(\mathcal{A})) \cong E(\mathbb{K})^{\oplus d}$.

Note that while the theorem applies to many interesting algebras, the hypotheses of the theorem are nevertheless quite restrictive. If $\mathcal{A}$ is the homogeneous coordinate ring of a smooth projective curve X, then the assumption that $\mathcal{A}$ has finite global dimension implies that X has genus zero. Indeed, the Hodge structures that appear in the theorem are of "Tate" type, meaning that they are built from tensor powers of the Hodge structure on $\mathsf{H}^2(\mathbb{P}^1)$; hence the noncommutative varieties in question have a "rational" flavour.

Before we go on, let us spell out the technical conditions that appeared in the statement of the theorem:

Definition 2.2. $\mathcal{A}$ has ***global dimension*** d if every finitely generated $\mathcal{A}$-module has a finite length resolution by finite rank projective $\mathcal{A}$-modules.

Definition 2.3. $\mathcal{A}$ is ***Koszul*** if the minimal resolution of $\mathbb{K}$ as a graded $\mathcal{A}$-module has the form

$$0 \longrightarrow \mathcal{A}(-d)^{\beta_d} \longrightarrow \cdots \longrightarrow \mathcal{A}(-2)^{\beta_2} \longrightarrow \mathcal{A}(-1)^{\beta_1} \longrightarrow \mathbb{K} \longrightarrow 0 \qquad (1)$$

where $\mathcal{A}(-k)$ denotes the graded $\mathcal{A}$-module obtained by shifting the grading on $\mathcal{A}$ down by k.

For a Koszul algebra with resolution (1), the Hilbert series always has the form

$$h_{\mathcal{A}}(t) = \frac{1}{1 - \beta_1 t + \cdots \pm \beta_{d-1} t^{d-1} \mp \beta_d t^d}$$

Thus our assumption on the Hilbert series in the Theorem 2.1 is equivalent to requiring that $\beta_d = 1$. Tabuada deals also with the more general case $\beta_d > 1$, but the statement of the theorem is slightly more subtle, which is why we restrict to the case $\beta_d = 1$ in this lecture.

2.1 Sketch of the proof

We now sketch the main ideas of the proof of Tabuada's theorem; we refer the reader to the original paper for the details [6].

2.1.1 The localization sequence

By definition, we have a short exact sequence of abelian categories:

$$0 \longrightarrow \mathsf{tors}(\mathcal{A}) \longrightarrow \mathsf{gr}(\mathcal{A}) \longrightarrow \mathsf{qgr}(\mathcal{A}) \longrightarrow 0,$$

i.e. $\mathsf{tors}(\mathcal{A})$ is a Serre subcategory and $\mathsf{qgr}(\mathcal{A})$ is the quotient, and this induces an exact sequence of the corresponding dg enhanced derived categories

$$0 \longrightarrow \mathsf{D}_{\mathsf{dg}}(\mathsf{tors}(\mathcal{A})) \longrightarrow \mathsf{D}_{\mathsf{dg}}(\mathsf{gr}(\mathcal{A})) \longrightarrow \mathsf{D}_{\mathsf{dg}}(\mathsf{qgr}(\mathcal{A})) \longrightarrow 0$$

By assumption, the invariant E is localizing, so it turns this short exact sequence of dg categories into a long exact sequence. It is therefore enough to understand $E(\mathsf{tors}(\mathcal{A}))$ and $E(\mathsf{gr}(\mathcal{A}))$, along with the induced morphism $E(\mathsf{tors}(\mathcal{A})) \to E(\mathsf{gr}(\mathcal{A}))$.

2.1.2 Graded modules versus graded vector spaces

Because the global dimension of $\mathcal{A}$ is finite, every finitely generated graded $\mathcal{A}$-module has a finite length projective resolution. Hence we have

$$\mathsf{D}_{\mathsf{dg}}(\mathsf{gr}(\mathcal{A})) \cong \mathsf{D}_{\mathsf{dg}}(\mathsf{gr}(\mathsf{proj}(\mathcal{A})))$$

where the category on the right hand side is the derived category formed from complexes of projective graded $\mathcal{A}$-modules. We note that every $\mathcal{M} \in \mathsf{gr}(\mathsf{proj}(\mathcal{A}))$ is isomorphic to a direct sum

$$\mathcal{M} \cong \bigoplus_{j=1}^{r} \mathcal{A}(k_j)$$

where $k_j \in \mathbb{Z}$. Thus $\mathcal{M}$ is completely determined by the shifts that appear. The same is true for graded vector spaces, so we conclude that there is a canonical bijection between isomorphisms classes of objects in the categories $\mathsf{gr}(\mathsf{proj}(\mathcal{A}))$ and $\mathsf{gr}(\mathbb{K})$. This suggests that $\mathsf{gr}(\mathsf{proj}(\mathcal{A}))$ and $\mathsf{gr}(\mathbb{K})$ might have the same invariants, but it is far from a proof since the morphisms in these two categories are quite different.

To get around this discrepancy, Tabuada uses the fact that any module $\mathcal{M}$ has a canonical filtration

$$\cdots \subset F^j\mathcal{M} \subset F^{j+1}\mathcal{M} \subset \cdots \mathcal{M}$$

where $F^j\mathcal{M}$ is the submodule generated by elements of degree at most j. Similarly, the category $\mathsf{gr}(\mathsf{proj}(\mathcal{A}))$ has an increasing filtration by subcategories

$$\mathsf{gr}(\mathsf{proj}(\mathcal{A}))_q \subset \mathsf{gr}(\mathsf{proj}(\mathcal{A}))$$

consisting of modules that are generated by their components of degree $[-q, q]$. By an inductive argument using these filtrations, and the hypotheses that E is localizing and cocontinuous, Tabuada establishes an isomorphism

$$E(\mathsf{gr}(\mathsf{proj}(\mathcal{A}))) \cong E(\mathsf{gr}(\mathbb{K})) \cong E\left(\coprod_{n\in\mathbb{Z}} \mod \mathbb{K}\right) \cong \bigoplus_{n\in\mathbb{Z}} E(\mathbb{K}).$$

2.1.3 Koszul duality

We now use Koszul duality. Recall that any Koszul algebra $\mathcal{A}$ is isomorphic to a quotient of the tensor algebra $T(\mathcal{A}_1)$ by the ideal generated by a set of quadratic relations

$$\mathcal{R} \subset \mathcal{A}_1 \otimes \mathcal{A}_1,$$

The ***Koszul dual*** algebra $\mathcal{A}^!$ is the quotient of the dual tensor algebra $T(\mathcal{A}_1^\vee)$ be the ideal generated by the annihilator

$$\mathcal{R}^\perp \subset \mathcal{A}_1^\vee \otimes \mathcal{A}_1^\vee.$$

Example 2.4. Suppose that $\mathcal{A}_1$ is two-dimensional with basis x, y, and

$$\mathcal{R} = \langle xy - yx \rangle \subset \mathcal{A}_1 \otimes \mathcal{A}_1$$

so that

$$\mathcal{A} = T(\mathcal{A}_1)/(\mathcal{R}) = \mathcal{S}ym(\mathcal{A}_1) \cong \mathbb{K}[x, y]$$

is the polynomial ring in x and y. The annihilator of $\mathcal{R}$ is given by

$$\mathcal{R}^\perp = \left\langle (x^\vee)^2, (y^\vee)^2, x^\vee y^\vee + y^\vee x^\vee \right\rangle \subset \mathcal{A}_1^\vee \otimes \mathcal{A}_1^\vee$$

where $x^\vee$ and $y^\vee$ are the dual basis to x, y. Hence the Koszul dual is the exterior algebra $\mathcal{A}^! = \wedge^\bullet \mathcal{A}_1^\vee$. A similarly duality between symmetric algebras and exterior algebras persists in all dimensions. □

Koszul duality gives an equivalence between subcategories of $\mathsf{D}_{\mathsf{dg}}(\mathsf{gr}(\mathcal{A}))$ and $\mathsf{D}(\mathsf{gr}(\mathcal{A}^!))$. Under this equivalence, the object $\mathbb{K}(i) \in \mathsf{D}_{\mathsf{dg}}(\mathsf{gr}(\mathcal{A}))$ corresponds to the object $\mathcal{A}^!(i)[-i] \in \mathsf{D}_{\mathsf{dg}}(\mathsf{gr}(\mathcal{A})^!)$, and this leads to an equivalence

$$\mathsf{D}_{\mathsf{dg}}(\mathsf{tors}(\mathcal{A})) \cong \mathsf{D}_{\mathsf{dg}}(\mathsf{gr}(\mathsf{proj}(\mathcal{A}^!))).$$

Applying the argument of Section 2.1.2 to $\mathcal{A}^!$ instead of $\mathcal{A}$, we find that

$$E(\mathsf{tors}(\mathcal{A})) \cong E(\mathsf{gr}(\mathsf{proj}(\mathcal{A}^!))) \cong \bigoplus_{n\in\mathbb{Z}} E(\mathbb{K}).$$

2.1.4 Reduce to the case of K-theory

From the calculations above, we conclude that localization triangle has the form

$$\bigoplus_{n\in\mathbb{Z}} E(\mathbb{K}) \xrightarrow{M} \bigoplus_{n\in\mathbb{Z}} E(\mathbb{K}) \longrightarrow E(\mathsf{qgr}(\mathcal{A}))$$

It therefore suffices to determine M. It follows from Tabuada's construction of the category of noncommutative motives [5] that it is sufficient to calculate M in the case of K-theory, i.e. $E(-) = \mathsf{K}_0(-)$. More precisely, there is a universal localizing invariant of dg categories which takes values in a triangulated category of "noncommutative motives". All other localizing invariants factor through the category of motives, and morphisms from the motive of $\mathbb{K}$ to the motive of an arbitrary dg category $\mathcal{B}$ are given by the K-theory $\mathsf{K}_0(\mathcal{B})$.

When $E(-) = \mathsf{K}_\bullet(-)$, we have an isomorphism

$$E(\mathbb{K}) = \mathsf{K}_0(\mathbb{K}) = \mathbb{Z}$$

given by the Euler characteristic. Hence M is described by a column-finite integer matrix with coefficients in $\mathbb{Z}$.

Let us make the identification $\mathsf{K}_0(\mathsf{gr}(\mathbb{K})) = \mathbb{Z}[t, t^{-1}]$, where t^j corresponds to the K-theory class of the graded module $\mathbb{K}(j)$. Since the inclusion $\mathsf{tors}(\mathcal{A}) \subset \mathsf{gr}(\mathcal{A})$ is compatible with grading shifts, the localization sequence must have the form

$$\mathbb{Z}[t, t^{-1}] \xrightarrow{\cdot M} \mathbb{Z}[t, t^{-1}] \longrightarrow \mathsf{K}^0(\mathsf{D}_{\mathsf{dg}}(\mathsf{qgr}(\mathcal{A})))$$

for some element $M \in \mathbb{Z}[t, t^{-1}]$. But the Koszul resolution (1) tells us that we have the following identity in K-theory:

$$[\mathbb{K}] = [\mathcal{A}] - \beta_1[\mathcal{A}(-1)] + \cdots \pm [\mathcal{A}(-d)].$$

We therefore have

$$M = 1 - \beta_1 t + \beta_2 t^2 - \cdots \pm \beta_d t^d.$$

Since we assume $\beta_d = 1$, it follows that

$$\mathsf{K}_0(\mathsf{D}(\mathsf{qgr}(\mathcal{A}))_{dg}) = \mathbb{Z}[t, t^{-1}]/M\mathbb{Z}[t, t^{-1}] = \mathbb{Z} \oplus \mathbb{Z}t \oplus \cdots \oplus \mathbb{Z}t^{d-1}$$

Therefore $\mathsf{K}^0(\mathsf{D}(\mathsf{qgr}(\mathcal{A}))_{dg}) \cong \mathsf{K}^0(\mathbb{K})^{\oplus d}$, as desired.

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