Noncommutative Hodge Theory
Lecture 5: Invariance properties of cohomology and the Gysin triangle

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January 22, 2018

Abstract

We recall the definition of the derived category of modules over a dg category, and use it to formulate natural invariance properties for Hochschild and cyclic homology. This leads to a noncommutative analogue of the classical Gysin sequence, which relates the cohomology of a submanifold to that of its complement.

Contents

1 Motivation: the classical Gysin sequence

Let X be a smooth variety, let $i : \mathsf{Z} \hookrightarrow \mathsf{X}$ be the inclusion of a smooth closed subvariety of codimension c, and let $j : \mathsf{U} \hookrightarrow \mathsf{X}$ be the inclusion of the open complement $\mathsf{U} = \mathsf{X} \setminus \mathsf{Z}$. Then the cohomology groups of X, U and Z are related by the Gysin long exact sequence in cohomology:

$$\cdots \longrightarrow \mathsf{H}^{i-2c}_{\mathsf{dR}}(\mathsf{Z}) \xrightarrow{i_*} \mathsf{H}^{i}_{\mathsf{dR}}(\mathsf{X}) \xrightarrow{j^*} \mathsf{H}^{i}_{\mathsf{dR}}(\mathsf{U}) \xrightarrow{\mathrm{Res}} \mathsf{H}^{i+1-2c}_{\mathsf{dR}}(\mathsf{Z}) \longrightarrow \cdots \quad (1)$$

Here i_* is the pushforward on cohomology obtained via Poincaré duality from the pushforward $i_* : \mathsf{H}_\bullet(\mathsf{Z}) \to \mathsf{H}_\bullet(\mathsf{Z})$ on homology, and Res is a certain "residue map". For instance, in the case when X is a curve and Z is a collection of points, this is induced by the usual reside operation on meromorphic one-forms.

The Gysin sequence is compatible with the (mixed) Hodge structures in a suitable sense, and therefore facilitates their computation. In this lecture, we discuss a similar result that aids calculations of the noncommutative Hodge structures of arbitrary dg categories. The idea is to obtain the sequence (1) via the periodic cyclic homology of the sequence of categories

$$\mathsf{Perf}(\mathsf{X})_{\mathsf{Z}} \longrightarrow \mathsf{Perf}(\mathsf{X}) \longrightarrow \mathsf{Perf}(\mathsf{U})$$

where $\mathsf{Perf}(\mathsf{X})_{\mathsf{Z}}$ denotes the category of perfect complexes on X whose cohomology sheaves are supported on Z, i.e. the perfect complexes that become trivial when restricted to U. This sequence of categories is "exact" in a certain sense, which implies that it induces a long exact sequence on periodic cyclic homology.

To make the notion of exactness precise, we will need to delve further into the theory of dg categories and their modules. We will mostly be following the papers [2, 6, 12], to which we refer the reader for more detail and comprehensive references.

Throughout these notes $\mathbb{K}$ is a field of characteristics zero, although the constructions can often be made more general. The symbol $\otimes := \otimes_{\mathbb{K}}$ denotes the tensor product over $\mathbb{K}$. We will use cohomological grading conventions for complexes. We denote by $\mathsf{Ch}(\mathbb{K})$ the dg category of cochain complexes of $\mathbb{K}$-modules (i.e. dg $\mathbb{K}$-modules), and denote by $\mathsf{Hom}^\bullet_{\mathsf{Ch}(\mathbb{K})}(\mathcal{M}, \mathcal{N})$ the complex of morphisms between two complexes $\mathcal{M}, \mathcal{N} \in \mathsf{Ch}(\mathbb{K})$.

2 Modules over dg categories

2.1 The dg category of dg categories

Recall that a ***dg category*** is a category $\mathcal{A}$ enriched over $\mathsf{Ch}(\mathbb{K})$, so that for any pair of objects $x, y \in \mathcal{A}$, the morphisms from x to y form a dg $\mathbb{K}$-module $\mathsf{Hom}^\bullet_{\mathcal{A}}(x, y)$ and the compositions are bilinear morphisms of complexes. If $\mathcal{A}$ and $\mathcal{B}$ are dg categories, a dg functor $F : \mathcal{A} \to \mathcal{B}$ is a functor whose action on the morphism spaces is given by cochain maps. We denote by $\mathsf{dgCat}(\mathbb{K})$ the two-category whose objects are essentially small dg categories, whose morphisms are dg functors, and whose two-morphisms are natural transformations.

We now record some basic structural features of $\mathsf{dgCat}(\mathbb{K})$:

- $\mathsf{dgCat}(\mathbb{K})$ has internal homs, i.e. for any pair of dg categories $\mathcal{A}$ and $\mathcal{B}$ the category $\mathsf{Hom}_{\mathsf{dgCat}(\mathbb{K})}(\mathcal{A}, \mathcal{B})$ of functors $\mathcal{A} \to \mathcal{B}$ and their natural transformations is itself a dg category.

  The key point is that if $F, G : \mathcal{A} \to \mathcal{B}$ are two dg functors, then the set of natural transformations $\eta : F \Rightarrow G$ forms a complex in a canonical way. Namely, such a transformation η is, by definition, an assignment of a morphism $\eta_x : F(x) \to G(x)$ for every object $x \in \mathcal{A}$, in a manner that is compatible with compositions, so that the natural transformations form a subcomplex $\mathcal{H}om^\bullet(F,G) \subset \prod_{x\in\mathcal{A}} \mathsf{Hom}^\bullet_{\mathcal{B}}(F(x),G(x))$. Note that in order to make sense of the direct product, we are implicitly using the fact that $\mathcal{A}$ is essentially small in order to avoid set-theoretic issues.

- There is a natural tensor product

$$- \otimes - : \mathsf{dgCat}(\mathbb{K}) \times \mathsf{dgCat}(\mathbb{K}) \to \mathsf{dgCat}(\mathbb{K})$$

making $\mathsf{dgCat}(\mathbb{K})$ into a symmetric monoidal category. If $\mathcal{A},\mathcal{B} \in \mathsf{dgCat}(\mathbb{K})$, then $\mathcal{A}\otimes\mathcal{B}$ is the dg category whose objects are pairs $(x,y) \in \mathcal{A}\times\mathcal{B}$ and whose morphisms $(x,y) \to (x',y')$ are given by the tensor product of complexes $\mathsf{Hom}^\bullet_{\mathcal{A}}(x,x') \otimes \mathsf{Hom}^\bullet_{\mathcal{B}}(y,y')$.

For instance, if $\mathcal{A}$ and $\mathcal{B}$ are dg algebras, viewed as one-object dg categories, then $\mathcal{A}\otimes\mathcal{B}$ is the one-object dg category corresponding to the usual tensor product of dg algebras.

- If $\mathcal{A},\mathcal{B},\mathcal{C} \in \mathsf{dgCat}(\mathbb{K})$ we have a tensor-hom adunction

$$\mathsf{Hom}_{\mathsf{dgCat}(\mathbb{K})}(\mathcal{A}\otimes\mathcal{B},\mathcal{C}) \cong \mathsf{Hom}_{\mathsf{dgCat}(\mathbb{K})}(\mathcal{A},\mathcal{H}om(\mathcal{B},\mathcal{C})) \qquad (2)$$

2.2 Modules over dg categories

There is a natural notion of module over a dg category. To get a feeling for the definition, let us first consider the case in which $\mathcal{A}$ is a dg algebra, or equivalently a dg category with a single object. We have several equivalent ways of defining what it means for $\mathcal{M}$ to be a (right) dg module over $\mathcal{A}$:

1. As a space with an $\mathcal{A}$ action: $\mathcal{M}$ is a complex of $\mathbb{K}$-modules equipped with a map of cochain complexes $\mathcal{M}\otimes\mathcal{A} \to \mathcal{M}$ that defines an associative right action of $\mathcal{A}$ on $\mathcal{M}$.

2. As an algebra homomorphism: $\mathcal{M}$ is a complex of $\mathbb{K}$-modules that is equipped with a homomorphism $\mathcal{A}^{\mathsf{op}} \to \mathsf{End}_{\mathbb{K}}(\mathcal{M})$ of dg algebras.

3. As a functor: $\mathcal{M}$ is a dg functor $\mathcal{A}^{\mathsf{op}} \to \mathsf{Ch}(\mathbb{K})$ from the one-object category $\mathcal{A}^{\mathsf{op}}$ to the category of cochain complexes of $\mathbb{K}$-modules. The functor sends the object $*$ to the complex $\mathcal{M} \in \mathsf{Ch}(\mathbb{K})$ and the action of the functor on morphisms gives the module structure in the above senses.

This last version generalizes to a succint definition of a module over an arbitrary dg category:

Definition 2.1. Let $\mathcal{A} \in \mathsf{dgCat}(\mathbb{K})$ be a dg category. A ***right $\mathcal{A}$-module*** is a dg functor $\mathcal{M} : \mathcal{A}^{\mathsf{op}} \to \mathsf{Ch}(\mathbb{K})$. The ***dg category of right $\mathcal{A}$-modules*** is the functor dg category

$$\mathsf{Ch}(\mathcal{A}) := \mathcal{H}om(\mathcal{A}^{\mathsf{op}}, \mathsf{Ch}(\mathbb{K})).$$

Similarly, one can make sense of left modules as functors $\mathcal{A} \to \mathsf{Ch}(\mathbb{K})$.

By the tensor-hom adjunction (2), we see that a right dg module over $\mathcal{A}$ consists of a chain complex of vector spaces $\mathcal{M}_x^\bullet \in \mathsf{Ch}(\mathbb{K})$ for each object $x \in \mathcal{A}$, together with a morphism of complexes

$$\mathcal{M}_y^\bullet \otimes \mathsf{Hom}_{\mathcal{A}}^\bullet(x,y) \to \mathcal{M}_x^\bullet$$

for each pair of objects $x, y \in \mathcal{A}$, compatible with compositions. If $\mathcal{M}$ and $\mathcal{N}$ are dg $\mathcal{A}$-modules, then elements of the morphism complex $\mathsf{Hom}^\bullet_{\mathsf{Ch}(\mathcal{A})}(\mathcal{M},\mathcal{N})$ are given by collections of elements $\eta_x \in \mathsf{Hom}^\bullet_{\mathsf{Ch}(\mathbb{K})}(\mathcal{M}_x^\bullet,\mathcal{N}_x^\bullet)$ for $x \in \mathcal{A}$, such that the following diagram commutes for all pairs $x, y \in \mathcal{A}$:

$$\begin{array}{ccc} \mathcal{M}_y^\bullet \otimes \mathsf{Hom}_{\mathcal{A}}^\bullet(x,y) & \longrightarrow & \mathcal{M}_x^\bullet \\ \downarrow {\scriptstyle \eta_y \otimes 1} & & \downarrow {\scriptstyle \eta_x} \\ \mathcal{N}_y^\bullet \otimes \mathsf{Hom}_{\mathcal{A}}^\bullet(x,y) & \longrightarrow & \mathcal{N}_x^\bullet \end{array}$$

We emphasize that in the definition of elements $\eta \in \mathsf{Hom}^\bullet_{\mathsf{Ch}(\mathcal{A})}(\mathcal{M},\mathcal{N})$, the constituent maps $\eta_x : \mathcal{M}_x \to \mathcal{N}_x$ are not required to be morphisms of cochain complexes, a condition which is instead equivalent to requiring that η be a degree zero cocycle.

Definition 2.2. If $\mathcal{M}$ and $\mathcal{N}$ are dg $\mathcal{A}$-modules, a ***morphism from $\mathcal{M}$ to $\mathcal{N}$*** is a zero-cocycle

$$\eta \in \mathsf{Z}^0\mathsf{Hom}^\bullet_{\mathsf{Ch}(\mathcal{A})}(\mathcal{M},\mathcal{N}).$$

We say that a morphism η is a ***quasi-isomorphism*** if the corresponding cochain maps $\eta_x : \mathcal{M}_x^\bullet \to \mathcal{N}_x^\bullet$ are quasi-isomorphisms for all objects $x \in \mathcal{A}$.

We have a subcategory $\mathsf{Z}^0\mathsf{Ch}(\mathcal{A}) \subset \mathsf{Ch}(\mathcal{A})$ with the same objects, and the morphisms given by the zero-cocycles as above. The derived category of $\mathcal{A}$ is then obtained by inverting the quasi-isomorphisms:

Definition 2.3. The ***derived category*** $\mathsf{D}(\mathcal{A})$ is the localization of $\mathsf{Z}^0\mathsf{Ch}(\mathcal{A})$ at the class of quasi-isomorphisms.

The definition above makes it difficult to do practical calculations in $\mathsf{D}(\mathcal{A})$ (and also causes some set-theoretic complications with the definition of the morphisms). One wants instead to know that morphisms in $\mathsf{D}(\mathcal{A})$ and functors $\mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathsf{B})$ can be computed using appropriate resolutions, extending the constructions used to define the derived functors $\mathsf{Ext}^\bullet(-)$ and $\mathsf{Tor}_\bullet(-)$ of classical homological algebra. This is typically approached by putting an appropriate Quillen model structure on $\mathsf{Ch}(\mathcal{A})$, for which the weak equivalences are the quasi-isomorphisms; see, e.g. [6, Section 3.2] for a discussion of this approach and several relevant references.

2.3 Tensor products

If $\mathcal{M}$ is a right $\mathcal{A}$-module and $\mathcal{N}$ is a left $\mathcal{A}$-module, there is a tensor product $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N} \in \mathsf{Ch}(\mathbb{K})$; see, e.g. [2, C.3]. It is defined as the quotient

$$\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N} := \left(\bigoplus_{x \in \mathcal{A}} \mathcal{N}_x \otimes_{\mathbb{K}} \mathcal{M}_x \right) \Big/ \sim \in \mathsf{Ch}(\mathbb{K})$$

where the equivalence relation $\sim$ is generated by the rule

$$(m_x \cdot a) \otimes_{\mathbb{K}} n_y \sim m_x \otimes_{\mathbb{K}} (a \cdot n_y) \tag{3}$$

for any objects $x, y \in \mathcal{A}$, any morphism $a \in \mathsf{Hom}_{\mathcal{A}}(y, x)$, and any module elements $m_x \in \mathcal{M}_x$ and $n_y \in \mathcal{N}_y$.

2.4 Functors between module categories

Observe that if $F : \mathcal{A} \to \mathcal{B}$ is a dg functor, then composition with F induces a natural pullback dg functor on the module categories:

$$F^* : \mathsf{Ch}(\mathcal{B}) \to \mathsf{Ch}(\mathcal{A})$$

Moreover, F defines a bimodule

$$\mathcal{A} \times \mathcal{B}^{\mathsf{op}} \to \mathsf{Ch}(\mathbb{K})$$

via the formula $(x, y) \mapsto \mathsf{Hom}_{\mathcal{B}}(y, F(x))$. As a result there is also a functor

$$F_! = - \otimes_{\mathcal{A}} \mathcal{B} : \mathsf{Ch}(\mathcal{A}) \to \mathsf{Ch}(\mathcal{B})$$

which sends a module $\mathcal{M} \in \mathsf{Ch}(\mathcal{A})$ to the module $F_! \mathcal{M} \in \mathsf{Ch}(\mathcal{B})$ which associates to each $y \in \mathcal{B}$ the complex

$$(F_! \mathcal{M})_y = \left(\bigoplus_{x \in \mathcal{A}} \mathcal{N}_x \otimes_{\mathbb{K}} \mathsf{Hom}_{\mathcal{B}}(y, F(x)) \right) / \sim$$

where $\sim$ is given by the tensor product relation (3) as in the previous section.

If $\eta : \mathcal{M} \to \mathcal{N}$ is a quasi-isomorphism of $\mathcal{B}$-modules, then the induced map $F^*\eta : F^*\mathcal{M} \to F^*\mathcal{N}$ is evidently a quasi-isomorphism of $\mathcal{A}$-modules, and hence we have an induced functor

$$RF^* = F^* : \mathsf{D}(\mathcal{B}) \to \mathsf{D}(\mathcal{A}).$$

on the derived categories. Moreover, the functor $F_!$ constructed above can be derived to obtain a functor

$$LF_! : \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{B}).$$

that is left adjoint to F^*; see, e.g., [2, C.12], generalizing the discussion in the case of dg algebras from [1, Chapter 10].

3 Invariants of dg categories

3.1 Morita morphisms and the homotopy category

Definition 3.1. Suppose $F : \mathcal{A} \to \mathcal{B}$ a dg functor. We say that F is a ***Morita morphism*** if $F^* : \mathsf{D}(\mathcal{B}) \to \mathsf{D}(\mathcal{A})$ is an equivalence of categories.

For many purposes, it is useful to think of Morita morphisms as corresponding to isomorphisms of the corresponding "noncommutative spaces". For instance, Morita morphisms induce isomorphisms on natural invariants of dg categories, such as Hochschild and cyclic homology.

As a result, it can be useful to pass to a localization of $\mathsf{dgCat}(\mathbb{K})$ in which the Morita morphisms become isomorphisms:

Definition 3.2. We denote by $\mathsf{Hmo}(\mathbb{K})$ the ***homotopy category of dg categories***, given by the localization of $\mathsf{dgCat}(\mathbb{K})$ at the class of Morita morphisms.

As with the construction of the derived category, the fact that $\mathsf{Hmo}(\mathbb{K})$ is well-behaved rests on the construction (due to Tabuada [7]) of a suitable Quillen model structure on $\mathsf{dgCat}(\mathbb{K})$, for which the weak equivalences are the Morita morphisms.

3.2 Localizing invariants

We now turn to the notion of a short exact sequence of dg categories

$$\mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{p} \mathcal{C}$$

and the corresponding long exact sequence in cohomology.

Definition 3.3. A ***short exact sequence of dg categories*** is a sequence

$$\mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{p} \mathcal{C}$$

in the homotopy category $\mathsf{Hmo}(\mathbb{K})$ such that i is the kernel of p, and p is the cokernel of i.

Rather then spelling out exactly what it means for i and p to be the kernel and cokernel, let us state the following result, which it to the more familiar notion of a Verdier quotient:

Theorem 3.4 (Keller [5, Section 4])**.** *A sequence $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ of morphisms in $\mathsf{Hmo}(\mathbb{K})$ is a short exact sequence if and only if it induces an embedding $\mathsf{D}(\mathcal{A}) \hookrightarrow \mathsf{D}(\mathcal{B})$ as a thick Serre subcategory, and identifies $\mathsf{D}(\mathcal{C})$ with the Verdier quotient $\mathsf{D}(\mathcal{B})/\mathsf{D}(\mathcal{A})$.*

The following key example of a short exact sequence is attributed to Thomason–Trobaugh [10, Section 5]; see also [9, Proposition 4.21]:

Theorem 3.5. *Let* X *be a quasi-compact, quasi-separated scheme, let* $\mathsf{U} \subset \mathsf{X}$ *be a quasi-compact open subset, and let* $\mathsf{Z} = \mathsf{X} \setminus \mathsf{U}$ *be the closed complement. Then the sequence*

$$\mathsf{Perf}(\mathsf{X})_{\mathsf{Z}} \longrightarrow \mathsf{Perf}(\mathsf{X}) \longrightarrow \mathsf{Perf}(\mathsf{U})$$

defines a short exact sequence of dg categories. Here $\mathsf{Perf}(\mathsf{X})$ *and* $\mathsf{Perf}(\mathsf{U})$ *are the categories of perfect complexes on* X *and* U*, respectively, and* $\mathsf{Perf}(\mathsf{X})_{\mathsf{Z}}$ *is the full subcategory consisting of perfect complexes whose cohomology sheaves are supported on* Z*.*

This result allows us to think of arbitrary short exact sequence of dg categories as corresponding to a "localization" of some noncommutative algebraic varieties. A localizing invariant of dg categories is one that turns such a localization of noncommutative varieties into a long exact sequence in cohomology. More precisely:

Definition 3.6. Let $\mathcal{T}$ be a triangulated category. A functor $E : \mathsf{dgCat}(\mathbb{K}) \to \mathcal{T}$ is a ***localizing invariant*** if

1. E sends Morita morphisms in $\mathsf{dgCat}(\mathbb{K})$ to isomorphisms in $\mathcal{T}$ (and hence factors through $\mathsf{Hmo}(\mathbb{K})$).
2. The induced functor $E : \mathsf{Hmo}(\mathbb{K}) \to \mathcal{T}$ sends short exact sequences to distinguished triangles, and this assignment is functorial for maps of short exact sequences.

Theorem 3.7 (Keller [4])**.** *The Hochschild and cyclic homology functors* $\mathsf{HH}_\bullet(-)$, $\mathsf{HC}_\bullet(-)$ *are localizing invariants taking values in the derived category* $\mathsf{D}(\mathbb{K})$*. Similarly the periodic cyclic homology* $\mathsf{HP}_\bullet(-)$ *is a localizing invariant taking values in the two-periodic derived category* $\mathsf{D}(\mathbb{K})_{\mathbb{Z}/2\mathbb{Z}}$*.*

In fact, Keller proves all the statements in Theorem 3.7 at once by proving the corresponding statement about the "mixed complex" in the sense of Kassel [3]. The latter is a convenient way to package the full data of the Hochschild complex equipped with the additional data of the Connes–Tsygan differential into a single algebraic gadget.

3.3 $\mathbb{A}^1$-homotopy invariance

Suppose that $\mathsf{Z} \subset \mathsf{X} \supset \mathsf{U}$ is a pair of a closed subscheme and its open complement, giving a short exact sequence of dg categories as in Theorem 3.5. Suppose further that X is smooth, so that Keller's global HKR theorem (Lecture 4, Section 3.5) gives isomorphisms $\mathsf{HP}_\bullet(\mathsf{X}) \cong \mathsf{H}^\bullet_{\mathsf{dR}}(\mathsf{X})$ and $\mathsf{HP}_\bullet(\mathsf{U}) \cong \mathsf{H}^\bullet_{\mathsf{dR}}(\mathsf{U})$. Applying Theorem 3.7, we obtain a long exact sequence

$$\cdots \to \mathsf{HP}_{\mathsf{ev}}(\mathsf{Perf}(\mathsf{X})_{\mathsf{Z}}) \to \mathsf{H}^{\mathsf{ev}}_{\mathsf{dR}}(\mathsf{X}) \to \mathsf{H}^{\mathsf{ev}}_{\mathsf{dR}}(\mathsf{U}) \to \mathsf{HP}_{\mathsf{odd}}(\mathsf{Perf}(\mathsf{X})_{\mathsf{Z}}) \to \cdots \quad (4)$$

This resembles a $\mathbb{Z}/2\mathbb{Z}$-graded version of the classical Gysin sequence (1), but there is a discrepancy: to get a perfect match, we would need to know that

$$\mathsf{HP}_\bullet(\mathsf{Perf}(\mathsf{X})_\mathsf{Z}) \cong \mathsf{HP}_\bullet(\mathsf{Perf}(\mathsf{Z})) \cong \mathsf{H}^\bullet_{\mathsf{dR}}(\mathsf{Z}) \tag{5}$$

and this is not immediately obvious. The difficulty is that the categories $\mathsf{Perf}(\mathsf{X})_\mathsf{Z}$ and $\mathsf{Perf}(\mathsf{Z})$ are genuinely different; the objects of the former are supported set-theoretically on Z, but their scheme-theoretic supports may be some larger infinitesimal thickenings of Z. However, one might expect such infinitesimal neighbourhoods to "retract" onto Z in some appropriate sense, so that topological invariants like $\mathsf{HP}_\bullet(-)$ end up being the same.

To make this precise, we need to recall the natural notion of homotopy of dg categories, as follows. If $\mathcal{A}$ is a dg category, we may extend its scalars to the polynomial ring $\mathbb{K}[t]$ by forming the tensor product of dg categories $\mathcal{A}[t] := \mathcal{A} \otimes \mathbb{K}[t]$. The inclusion $\mathbb{K} \to \mathbb{K}[t]$ induces a natural functor $\mathcal{A} \to \mathcal{A}[t]$. Notice that we have $\mathbb{K}[t] = \mathcal{O}(\mathbb{A}^1)$ where $\mathbb{A}^1$ is the affine line over $\mathbb{K}$. If $\mathcal{A}$ is the category of sheaves on some scheme X, then $\mathcal{A}[t]$ is the category of sheaves on $\mathsf{X} \times \mathbb{A}^1$, and the functor $\mathcal{A} \to \mathcal{A}[t]$ corresponds to the pullback along the projection $\mathsf{X} \times \mathbb{A}^1 \to \mathsf{X}$.

Definition 3.8. Let $\mathcal{T}$ be a triangulated category. A functor $E : \mathsf{dgCat}(\mathbb{K}) \to \mathcal{T}$ is $\mathbb{A}^1$***-homotopy invariant*** if for any dg category $\mathcal{A}$, the map $E(\mathcal{A}) \to E(\mathcal{A}[t])$ is an isomorphism.

Example 3.9. Periodic cyclic homology $\mathsf{HP}_\bullet(-)$ is $\mathbb{A}^1$-homotopy invariant when $\mathbb{K}$ is a field of characteristic zero. (This is a noncommutative analogue of the fact that the de Rham cohomology of a smooth complex variety only depends on the topology of its underlying manifold.) As observed in [8, Section 3], the homotopy invariance follows from a more general Künneth-type isomorphism for periodic cyclic homology, which was established by Kassel in the case of ordinary algebras [3]. Extended to dg categories, is gives an isomorphism

$$\mathsf{HP}_\bullet(\mathcal{A}[t]) \cong \mathsf{HP}_\bullet(\mathcal{A}) \otimes \mathsf{HP}_\bullet(\mathbb{K}[t]).$$

But $\mathsf{HP}_\bullet(\mathbb{K}[t]) \cong \mathsf{H}^\bullet_{\mathsf{dR}}(\mathbb{A}^1) \cong \mathbb{K}$ by the HKR theorem and the Poincaré lemma for polynomial differential forms on $\mathbb{A}^1$. □

Example 3.10. In contrast, Hochschild homology $\mathsf{HH}_\bullet(-)$ is *not* $\mathbb{A}^1$-homotopy invariant. Indeed, it is easy to see that this fails already in the commutative case. By the HKR theorem, we have $\mathsf{HH}_p(\mathbb{K}) = 0$ for $p \neq 0$. Meanwhile $\mathsf{HH}_1(\mathbb{K}[t]) \cong \Omega^1(\mathbb{A}^1) = \mathbb{K}[t]\,\mathrm{d}t \neq 0$. □

Tabuada and Van den Bergh have recently shown that the $\mathbb{A}^1$-homotopy invariance property is enough to establish the desired isomorphism (5) and hence recover the classical Gysin sequence (1) from the localization sequence (4). In fact, they establish a Gysin sequence for any localizing $\mathbb{A}^1$-homotopy invariant:

Theorem 3.11 (Tabuada–Van den Bergh [9])**.** *Let X a smooth scheme over a field $\mathbb{K}$ of characteristic zero. Let $i : \mathsf{Z} \hookrightarrow \mathsf{X}$ be the inclusion of a smooth* *closed subscheme and* $j : \mathsf{U} \hookrightarrow \mathsf{X}$ *the inclusion of its complement. If a functor* $E : \mathsf{dgCat}(\mathbb{K}) \to \mathcal{T}$ *is localizing and* $\mathbb{A}^1$*-homotopy invariant, then there is a canonical exact triangle*

$$E(\mathsf{Perf}_{\mathsf{dg}}(\mathsf{Z})) \xrightarrow{E(i_*)} E(\mathsf{Perf}_{\mathsf{dg}}(\mathsf{X})) \xrightarrow{E(j^*)} E(\mathsf{Perf}_{\mathsf{dg}}(\mathsf{U})) \longrightarrow E(\mathsf{Perf}_{\mathsf{dg}}(\mathsf{Z}))[1]$$

in the triangulated category $\mathcal{T}$.

Example 3.12. Let $\mathsf{X} = \mathsf{Spec}(\mathbb{K}[t]) = \mathbb{A}^1$ and let $\mathsf{Z} = \mathsf{Spec}(\mathbb{K}) \subset \mathsf{X}$ be the origin. Let $\mathsf{U} = \mathsf{Spec}(\mathbb{K}[t, t^{-1}])$. Using the Gysin sequence we find

$$E(\mathbb{K}[t, t^{-1}]) \cong E(\mathbb{K}) \oplus E(\mathbb{K})[1]$$

and more generally, $E(\mathcal{A}[t, t^{-1}]) = E(\mathcal{A}) \oplus E(\mathcal{A})[1]$. This matches expectations since the complex manifold underlying U is $\mathbb{C} \setminus \{0\}$, which has the homotopy type of a circle. □

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