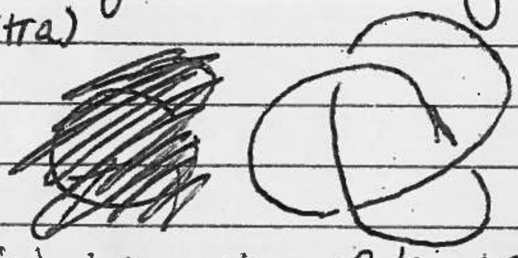
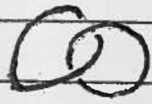
Noah White - Knot Invariants

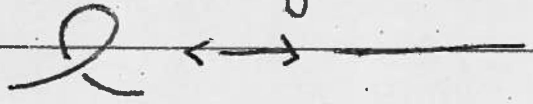
1. Knots

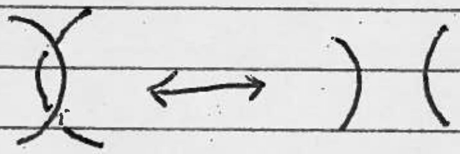
Def: A <u>knot</u> is a subset $K \subseteq \mathbb{R}^3$ s.t. K is homeomorphic to S^1 (+ extra conditions)
Say 2 knots are equivalent if they are isotopic (+ extra)

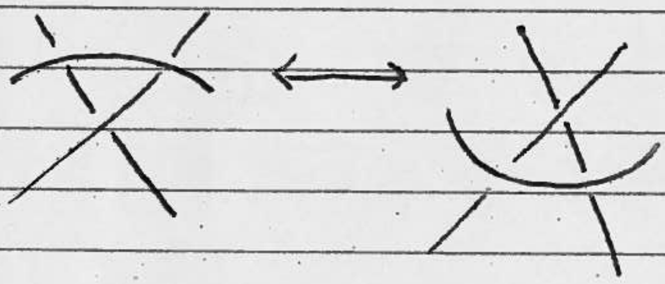
Ex: $\mathcal{U}$ =  or 

Def: A <u>link</u> is a disjoint union of knots. 
Also, <u>oriented</u> knot/link

Thm: (Reidemeister) Two diagrams represent equivalent knots (links) if they are related by the following moves.

① 

② 

③ 

Aim: Construct invariants

Ex: ① L a link, $\mu(L)$ = # connected components
② K a knot, $c(K) = \min\{\#$ crossings for D, D a diagram of $K\}$.

③ Jones Polynomial

2. Ribbon Categories

- $\mathcal{U}_q(\mathfrak{g})$-mod (quantized universal enveloping algebra of Lie algebra $\mathfrak{g}$)
- H-mod, H a Hopf algebra
- $\mathbb{C}G$-mod, G a finite group
- $\mathrm{Vect}^{fin}_{\mathbb{C}}$

Def: A category $\mathcal{C}$ is <u>monoidal</u> if it has a bifunctor $\_\otimes\_ : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ and there exists a unit object $\mathbb{1}\in\mathcal{C}$ s.t. $\mathbb{1}\otimes V\cong V\cong V\otimes\mathbb{1}$ for all objects $V\in\mathcal{C}$. (+ extra)

In $\mathrm{Vect}^{fin}_{\mathbb{C}}$, $V\otimes W\cong W\otimes V$. A monoidal category $\mathcal{C}$ is called <u>braided</u> if there exist natural isomorphisms $V\otimes W\cong W\otimes V$.

$$\sigma : V\otimes W\to W\otimes V$$

In $\mathrm{Vect}^{fin}_{\mathbb{C}}$, $\sigma^2 = \mathrm{Id}$ since $\sigma = \text{Flip}$. Not true in general.

In $\mathrm{Vect}^{fin}_{\mathbb{C}}$, $V^*\otimes V\xrightarrow{ev.}\mathbb{C}$

$\mathbb{C}\xrightarrow[i]{coev.}V\otimes V^*$

Def: A monoidal category is <u>rigid</u> if there exist duals for each object and natural isom: $V^*\otimes V\xrightarrow{ev_V}\mathbb{1}$ and $\mathbb{1}\xrightarrow{i_V}V\otimes V^*$

Have
$$\begin{tikzcd} \cancel{\mathbb{1}\otimes} V \arrow[r, "i_v\otimes id"] \arrow[d, equal] & V\otimes V^*\otimes V \arrow[dl, "id\otimes ev_v"] \\ V & \end{tikzcd}$$

If $V^{**}\overset{\delta_V}{\cong}V$, $\delta_{V\otimes W}=\delta_V\otimes\delta_W$ in $\mathrm{Vect}^{fin}_{\mathbb{C}}$. A rigid monoidal category is <u>balanced</u> if we have natural isomorphisms $\delta_V : V\to V^{**}$ s.t.

$$\delta_{V\otimes W}=\delta_V\otimes\delta_W.$$

Def. A <u>ribbon category</u> is a balanced, rigid, braided monoidal category.

<u>Ribbon</u>: · $\otimes$ tensor

· braiding $V\otimes W \xrightarrow{\sim} W\otimes V$

· duals $V \xrightarrow{i\otimes id} V\otimes V^*\otimes V \xrightarrow{id\otimes e} V$, $\ \ V = V$

· $\delta_V : V \xrightarrow{\sim} V^{**}$ s.t. $\delta_{V\otimes W} = \delta_V \otimes \delta_W$

In any rigid, braided monoidal category,

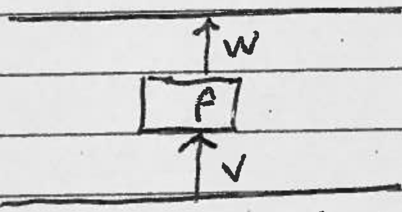
$$V^{**} \xrightarrow{i\otimes id} V\otimes V^*\otimes V^{**} \xrightarrow{id\otimes c} V\otimes V^{**}\otimes V^* \xrightarrow{id\otimes e} V$$

$\psi_V : V^{**} \xrightarrow{\sim} V$

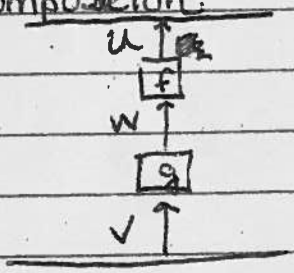
$\theta_V : \psi_V \circ \delta_V : V \to V$ the <u>twisting</u> of $\mathcal{C}$

3. Graphical Calculus

$f: V \to W$ instead replace objects with directed edges, morphisms with boxes.

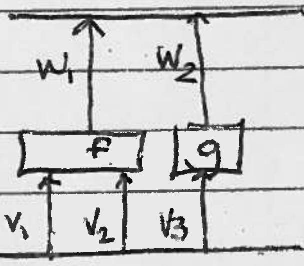


- <u>Composition</u>: $f\circ g: V \to W \to U$.

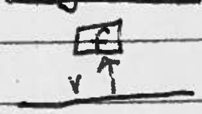


- Tensor products

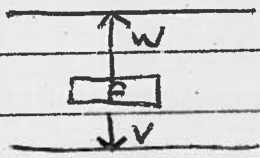
$f\otimes g: \underbrace{V_1\otimes V_2}_{f}\otimes \underbrace{V_3}_{g} \to W_1\otimes W_2$



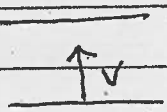
- <u>Identity object</u> $f: V \to \mathbb{1}$.



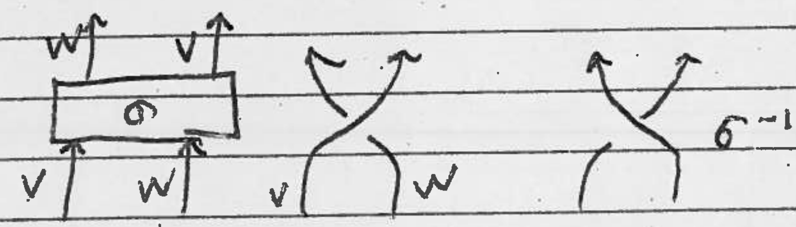
- Duals $f: V^* \to W$



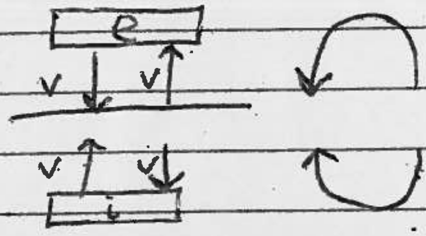
- <u>Identity morphism</u>:



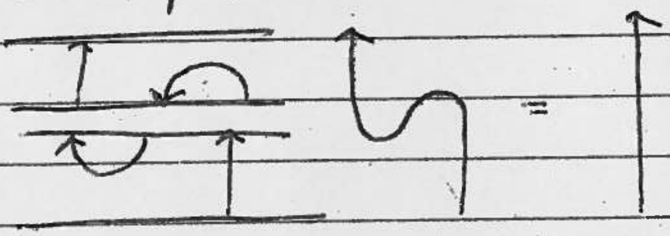
$\underline{\sigma}$



$\underline{e}$



$\underline{i}$

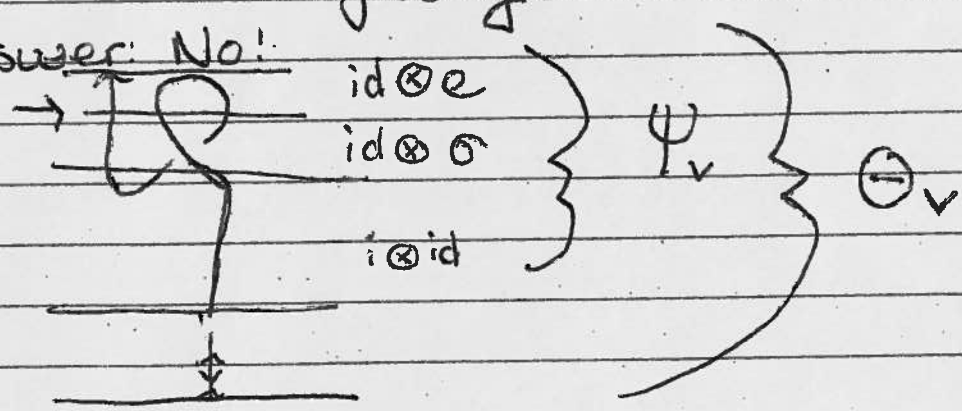


$(id \otimes e) \circ (i \otimes id) = id$

Can write Reidemeister relations this way, prove using category axioms.

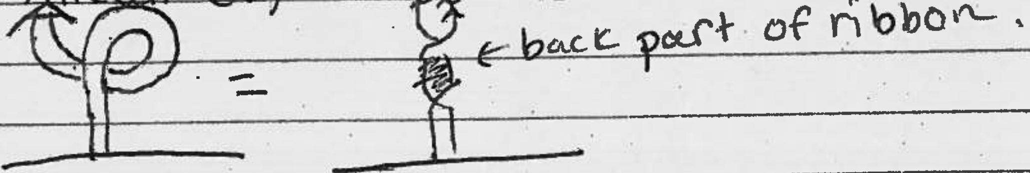
Question: Do these diagrams represent morphisms unambiguously?

Answer: No!



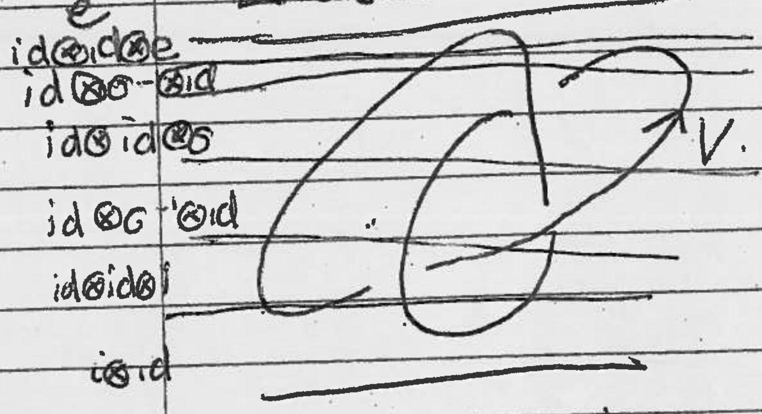
not identity but would by by Reidemeister

Fix: Thicken objects to ribbon



Then answer is yes.

$\mathrm{End}(\mathbb{1}) = \mathbb{C}$



Thicken, then get $\mathrm{End}(\mathbb{1})$ element.

Set $\mathcal{C} = U_q(\underline{sl_2})$-mod.

$V = V(1) = \mathbb{C}^2$

If multiply out as matrices, get Jones polynomial