Noncommutative Hodge Theory Lecture 4: Cyclic homology and the noncommutative Hodge filtration

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Abstract

We introduce the Connes–Tsygan differential on the Hochschild complex, which gives the (periodic) cyclic homology. In the commutative case, this recovers the de Rham cohomology, equipped with a variant of the Hodge filtration.

1 Motivation

In the previous lecture we saw the Hochschild–Kostant–Rosenberg (HKR) isomorphism:

Theorem 1.1 (HKR)**.** *If* X *is a smooth affine variety, then*

$$\mathsf{HH}_k(\mathcal{O}(\mathsf{X})) \cong \Omega^k(\mathsf{X}).$$

Note that $\Omega^\bullet(\mathsf{X})$ carries the de Rham differential, and so far this structure has not appeared in the context of Hochschild homology. In this lecture, we review the construction of an additional differential B on the Hochschild homology $\mathsf{HH}_\bullet(\mathcal{A})$ for an arbitrary associative algebra $\mathcal{A}$, which originates in the works of Connes [1], Loday–Quillen [7] and Tsygan [9]. If $\mathcal{A}$ is the algebra of functions on a smooth affine variety, then B corresponds to the de Rham differential under the HKR isomorphism. Hence the resulting homology, known as (periodic) cyclic homology, gives a replacement for de Rham cohomology in the noncommutative setting. The lecture was based primarily on [2, 7]; there are many other useful references, including [4, 6, 8].

We will focus on the algebraic aspects of the construction, but before we start let us make a brief remark about the topological interpretation. Recall that when $\mathsf{X} = \mathsf{Spec}(\mathcal{A})$ is an affine variety, we can interpret the Hochschild homology $\mathsf{HH}_\bullet(\mathcal{A}) = \mathsf{Tor}_\bullet^{\mathcal{A}^e}(\mathcal{A},\mathcal{A})$ as the algebra of functions on the derived self intersection of the diagonal $\mathsf{X} \hookrightarrow \mathsf{X} \times \mathsf{X}$—also known as the "derived loop space" $\mathsf{Maps}(\mathsf{S}^1, \mathsf{X})$, a notion that can be made precise by looking at maps from a simplicial set presenting S^1 to the space X. As a result of the loop space interpretation, there is a "circle action by loop rotation", which corresponds algebraically to the cyclic rotation of the tensor factors in the Hochschild complex. This cyclic action induces the aforementioned differential B via the corresponding group (co)homology, so that the (periodic) cyclic homology can be viewed as a sort of S^1-equivariant version of the Hochschild homology.

2 Basic ingredients

The cyclic homology will be constructed by weaving together the Hochschild complex with a suitable action of the cyclic group. The basic ingredients are as follows.

2.1 The bar and Hochschild complexes

Recall that if $\mathcal{M}$ is an $\mathcal{A}$-bimodule, the bar resolution $\mathsf{Bar}_\bullet(\mathcal{A}, \mathcal{M})$ is a resolution of $\mathcal{M}$ as an $\mathcal{A}^{\mathsf{e}}$-module, given by

$$\mathsf{Bar}_k(\mathcal{A}, \mathcal{M}) = \mathcal{A}^{\otimes k+1} \otimes \mathcal{M}$$

with differential

$$\mathrm{b}'(a_0 \otimes \cdots \otimes a_k \otimes m) = \sum_{i=0}^{k} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes m$$

The Hochschild chain complex $\mathsf{C}_\bullet(\mathcal{A}, \mathcal{M}) = \mathcal{A} \otimes_{\mathcal{A}^{\mathsf{e}}} \mathsf{Bar}_\bullet(\mathcal{A}, \mathcal{M})$ is given by

$$\mathsf{C}_k(\mathcal{A}, \mathcal{M}) = \mathcal{A}^{\otimes k} \otimes \mathcal{M}$$

and differential

$$\mathrm{b} = \mathrm{b}' + (-1)^{k+1}\mathrm{t}$$

where

$$\mathrm{t}(a_0 \otimes \cdots \otimes a_k \otimes m) = a_1 \otimes \cdots \otimes a_k \otimes m a_0,$$

using the right module structure on $\mathcal{M}$.

In the case $\mathcal{M} = \mathcal{A}$, we obtain the Hochschild complex $\mathsf{C}_\bullet(\mathcal{A})$ and the bar resolution $\mathsf{Bar}_\bullet(\mathcal{A})$, which are very similar. They have the same terms but the differentials are different (b vs. b′.)

2.2 The cyclic action and its homology

Notice that the spaces $\mathcal{A}^{\otimes k}$, which appear in both $\mathsf{C}_\bullet(\mathcal{A})$ and $\mathsf{Bar}_\bullet(\mathcal{A})$ carry a natural action of the cyclic group $\mathsf{G}_k = \mathbb{Z}/k\mathbb{Z}$. The generator $\tau \in \mathsf{G}_k$ simply rotates the tensor factors via the formula

$$\tau(a_1 \otimes \cdots \otimes a_k) = (-1)^{k+1} a_2 \otimes \cdots \otimes a_k \otimes a_1.$$

Note that while the sign may at first seem to cause a problem, we do indeed have $\tau^k = ((-1)^{(k+1)})^k \mathrm{id} = \mathrm{id}$.

There is a standard complex that computes the group homology $\mathsf{H}_\bullet(\mathsf{G}_k, \mathcal{A}^{\otimes k})$, which we now recall. The group ring $\mathbb{Z}\mathsf{G}_k$ has an element $\eta_k = 1+\tau+\cdots+\tau^{k-1}$, and it is straightforward to check that

$$\eta_k(1-\tau) = (1-\tau)\eta_k = 0$$

in $\mathbb{Z}\mathsf{G}_k$. We also have the natural augmentation map $\mathbb{Z}\mathsf{G}_k \to \mathbb{Z}$. One can check that the resulting complex

$$\cdots \xrightarrow{\eta_k} \mathbb{Z}\mathsf{G}_k \xrightarrow{1-\tau} \mathbb{Z}\mathsf{G}_k \xrightarrow{\eta_k} \mathbb{Z}\mathsf{G}_k \xrightarrow{1-\tau} \mathbb{Z}\mathsf{G}_k \to \mathbb{Z},$$

extended periodically to the left, gives a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}\mathsf{G}_k$-module, so that the group homology of $\mathsf{H}_\bullet(\mathsf{G}_k, \mathcal{A}^{\otimes k}) = \mathsf{Tor}_\bullet^{\mathbb{Z}\mathsf{G}_k}(\mathbb{Z}, \mathcal{A}^{\otimes k})$ is the homology of the complex

$$\cdots \xrightarrow{\eta_k} \mathcal{A}^{\otimes k} \xrightarrow{1-\tau} \mathcal{A}^{\otimes k} \xrightarrow{\eta_k} \mathcal{A}^{\otimes k} \xrightarrow{1-\tau} \mathcal{A}^{\otimes k} \tag{1}$$

3 Cyclic homology

3.1 The cyclic bicomplex

One can verify that the the differentials on the Hochschild complex, the bar complex, and the cyclic group homology complex (1) are related by the following identities:

$$(\mathrm{id}-\tau)\mathrm{b}' = -\mathrm{b}(\mathrm{id}-\tau) \qquad\qquad \eta_{k-1}\mathrm{b}' = -\mathrm{b}\eta_k$$

As a result, these complexes can be stitched together to form the so-called ***cyclic bicomplex***:

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \mathcal{A} & \xrightarrow{\eta_1} & \mathcal{A} & \xrightarrow{0} & \mathcal{A} \\
 & & \uparrow{\scriptstyle \mathrm{b}'} & & \uparrow{\scriptstyle \mathrm{b}'} & & \uparrow{\scriptstyle \mathrm{b}} \\
\cdots & \longrightarrow & \mathcal{A}^{\otimes 2} & \xrightarrow{\eta_2} & \mathcal{A}^{\otimes 2} & \xrightarrow{\mathrm{id}-\tau} & \mathcal{A}^{\otimes 2} \\
 & & \uparrow{\scriptstyle \mathrm{b}'} & & \uparrow{\scriptstyle \mathrm{b}'} & & \uparrow{\scriptstyle \mathrm{b}} \\
\cdots & \longrightarrow & \mathcal{A}^{\otimes 3} & \xrightarrow{\eta_3} & \mathcal{A}^{\otimes 3} & \xrightarrow{\mathrm{id}-\tau} & \mathcal{A}^{\otimes 3} \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & \vdots & & \vdots & & \vdots
\end{array} \tag{2}$$

where the rows compute the group homology and the columns alternate between the Hochschild complex and the bar complex.

Definition 3.1. The ***cyclic homology*** $\mathsf{HC}_\bullet(\mathcal{A})$ is the homology of the total complex of the cyclic bicomplex (2).

Since the bar complex is a resolution, the odd degree columns of the cyclic bicomplex are exact. Hence the cyclic homology can be computed by a spectral sequence whose E^1 page has the following form:

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \mathsf{HH}_0(\mathcal{A}) & \longrightarrow & 0 & \longrightarrow & \mathsf{HH}_0(\mathcal{A}) \\
\\
\cdots & \longrightarrow & \mathsf{HH}_1(\mathcal{A}) & \longrightarrow & 0 & \longrightarrow & \mathsf{HH}_1(\mathcal{A}) \\
\\
\cdots & \longrightarrow & \mathsf{HH}_2(\mathcal{A}) & \longrightarrow & 0 & \longrightarrow & \mathsf{HH}_2(\mathcal{A}) \\
\\
 & & \vdots & & \vdots & & \vdots
\end{array}$$

Put differently, the E^1 page is isomorphic to $\mathsf{HH}_\bullet(\mathcal{A})[u^{-1}]$, where $\bullet$ indexes the rows and u is a formal variable of degree two that indexes the columns (using the cohomological grading). Hence we get a spectral sequence

$$\mathsf{HH}_\bullet(\mathcal{A})[u^{-1}] \Rightarrow \mathsf{HC}_\bullet(\mathcal{A})$$

Because the group homology complex is periodic to the left, we can extend the bicomplex above to be periodic to the right as well, in which case we obtain the ***periodic cyclic bicomplex***:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathcal{A} & \xrightarrow{\eta_1} & \mathcal{A} & \xrightarrow{0} & \mathcal{A} & \xrightarrow{\eta_1} & \cdots \\
 & & \uparrow{\scriptstyle b'} & & \uparrow{\scriptstyle b'} & & \uparrow{\scriptstyle b} & & \\
\cdots & \longrightarrow & \mathcal{A}^{\otimes 2} & \xrightarrow{\eta_2} & \mathcal{A}^{\otimes 2} & \xrightarrow{\mathrm{id}-\tau} & \mathcal{A}^{\otimes 2} & \xrightarrow{\eta_2} & \cdots \\
 & & \uparrow{\scriptstyle b'} & & \uparrow{\scriptstyle b'} & & \uparrow{\scriptstyle b} & & \\
\cdots & \longrightarrow & \mathcal{A}^{\otimes 3} & \xrightarrow{\eta_3} & \mathcal{A}^{\otimes 3} & \xrightarrow{\mathrm{id}-\tau} & \mathcal{A}^{\otimes 3} & \xrightarrow{\eta_3} & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array} \tag{3}$$

The rows of this bicomplex now give a variant of the group (co)homology, known as the ***Tate cohomology***.

Definition 3.2. The ***periodic cyclic homology*** $\mathsf{HP}_\bullet(\mathcal{A})$ is the total homology of the periodic cyclic bicomplex (3).

Note that since the bicomplex no longer sits in the third quadrant, some care is needed with the totalization: to form the periodic cyclic homology, one takes the direct product of the diagonals, rather than the direct sum. This gives a spectral sequence

$$\mathsf{HH}_\bullet(\mathcal{A})((u)) \Rightarrow \mathsf{HP}_\bullet(\mathcal{A})$$

using the formal Laurent series in u rather than polynomials in u^{-1}. Note also that because of the periodicity, there are really only two distinct periodic cyclic homology groups, namely the even- and odd-degree components $\mathsf{HP}_{\mathsf{ev}}(\mathcal{A})$ and $\mathsf{HP}_{\mathsf{odd}}(\mathcal{A})$.

3.2 The Connes–Tsygan differential

The bar complex $\mathsf{Bar}_\bullet(\mathcal{A})$ is acyclic, with contracting homotopy

$$\mathrm{h}(a_0 \otimes \cdots \otimes a_k) = 1 \otimes a_0 \otimes \cdots \otimes a_k.$$

We can use this homotopy to remove the odd degree columns in the (periodic) cyclic bicomplex.

More precisely, consider a local piece of the bicomplex where the homotopy acts:

$$\begin{array}{ccccc}
\mathcal{A}^{\otimes k} & \xrightarrow{\eta_k} & \mathcal{A}^{\otimes k} & & \\
 & & {\scriptstyle \mathrm{b}'}\uparrow \; \downarrow{\scriptstyle \mathrm{h}} & & \\
 & & \mathcal{A}^{\otimes (k+1)} & \xrightarrow{\mathrm{id}-\tau} & \mathcal{A}^{\otimes (k+1)}
\end{array}$$

It allows us to define the operator

$$\mathrm{B} = (\mathrm{id} - \tau) \cdot \mathrm{h} \cdot \eta_k : \mathcal{A}^{\otimes k} \to \mathcal{A}^{\otimes (k+1)}$$

We can then fold the original bicomplex into a simpler one whose total complex still computes the cyclic homology:

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{A} \\
 & & \uparrow & & \uparrow & & \uparrow{\scriptstyle \mathrm{b}} \\
\cdots & \longrightarrow & 0 & \longrightarrow & \mathcal{A} & \xrightarrow{\mathrm{B}} & \mathcal{A}^{\otimes 2} \\
 & & \uparrow & & \uparrow{\scriptstyle \mathrm{b}} & & \uparrow{\scriptstyle \mathrm{b}} \\
\cdots & \longrightarrow & \mathcal{A} & \xrightarrow{\mathrm{B}} & \mathcal{A}^{\otimes 2} & \xrightarrow{\mathrm{B}} & \mathcal{A}^{\otimes 3} \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & \vdots & & \vdots & & \vdots
\end{array} \tag{4}$$

3.3 The cyclic HKR theorem

We now specialize to the situation of the HKR theorem, and explain how to recover the de Rham differential.

Proposition 3.3. *Suppose that $\mathcal{A} = \mathcal{O}(\mathsf{X})$ is the algebra of functions on a smooth affine variety. Then the Hochschild–Kostant–Rosenberg isomorphism extends to a quasi-isomorphism from the bicomplex (4) to the following bicomplex built from the de Rham complex of X:*

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{O}(\mathsf{X}) \\
 & & \uparrow & & \uparrow & & \uparrow{\scriptstyle 0} \\
\cdots & \longrightarrow & 0 & \longrightarrow & \mathcal{O}(\mathsf{X}) & \xrightarrow{\mathrm{d_{dR}}} & \Omega^1(\mathsf{X}) \\
 & & \uparrow & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle 0} \\
\cdots & \longrightarrow & \mathcal{O}(\mathsf{X}) & \xrightarrow{\mathrm{d_{dR}}} & \Omega^1(\mathsf{X}) & \xrightarrow{\mathrm{d_{dR}}} & \Omega^2(\mathsf{X}) \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & \vdots & & \vdots & & \vdots
\end{array}$$

Proof. The proof is computational; one simply checks that the HKR map intertwines the differential B and the de Rham differential $\mathrm{d_{dR}}$. ☐

Note that the cyclic homology is therefore quite close to the de Rham cohomology of X, but it differs in low degrees because of the truncation that happens at the rightmost edge of the bicomplex. This can be remedied by passing to the periodic versions, where we obtain the following:

Corollary 3.4. *If X is a smooth affine variety then the periodic cyclic homology of the algebra $\mathcal{O}(\mathsf{X})$ is isomorphic to the even/odd degree de Rham cohomology:*

$$\mathsf{HP}_{\mathsf{ev}}(\mathcal{O}(\mathsf{X})) \cong \bigoplus_{k\geq 0} \mathsf{H}^{2k}_{\mathsf{dR}}(\mathsf{X}) \qquad \mathsf{HP}_{\mathsf{odd}}(\mathcal{O}(\mathsf{X})) \cong \bigoplus_{k\geq 0} \mathsf{H}^{2k+1}_{\mathsf{dR}}(\mathsf{X})$$

3.4 Categorical version

Just as one can extend the definition of Hochschild homology from associative algebras to dg categories, one can extend the definition of (periodic) cyclic homology. Keller's global categorical version of the HKR theorem works also for cyclic homology and its variants [4]. It gives an isomorphism

$$\mathsf{HP}_{\mathsf{ev/odd}}(\mathsf{Perf}(\mathsf{X})) \cong \mathsf{H}^{\mathsf{ev/odd}}_{\mathsf{dR}}(\mathsf{X})$$

for any smooth variety X (not just the affine ones treated in Corollary 3.4).

3.5 The Hodge filtration

As we have seen, the periodic cyclic homology really only has a $\mathbb{Z}/2\mathbb{Z}$-grading, given by the even and odd degree components. However, it does come from a bicomplex, so it has a natural *filtration*

$$\cdots \supset \mathsf{F}^p\mathsf{HP}_\bullet(\mathcal{A}) \supset \mathsf{F}^{p+1}\mathsf{HP}_\bullet(\mathcal{A}) \supset \cdots$$

induced by the Hochschild-to-periodic-cyclic spectral sequence. This is the ***non-commutative Hodge filtration***.

The convention in [3, 5] is to label the filtration on $\mathsf{HP}_{\mathsf{ev}}(\mathcal{A})$ by elements of $\mathbb{Z}$ and the filtration on $\mathsf{HP}_{\mathsf{odd}}(\mathcal{A})$ by elements of $\frac{1}{2}+\mathbb{Z}$. If X is a smooth affine variety, we get the filtrations

$$\mathsf{HP}_{\mathsf{ev}}(\mathcal{O}(\mathsf{X})) \cong \underbrace{\mathsf{H}^0_{\mathsf{dR}}(\mathsf{X}) \oplus \underbrace{\mathsf{H}^2_{\mathsf{dR}}(\mathsf{X}) \oplus \underbrace{\mathsf{H}^4_{\mathsf{dR}}(\mathsf{X}) \oplus \cdots}_{\mathsf{F}^{4/2}}}_{\mathsf{F}^{2/2}}}_{\mathsf{F}^{0/2}}$$

$$\mathsf{HP}_{\mathsf{odd}}(\mathcal{O}(\mathsf{X})) \cong \underbrace{\mathsf{H}^1_{\mathsf{dR}}(\mathsf{X}) \oplus \underbrace{\mathsf{H}^3_{\mathsf{dR}}(\mathsf{X}) \oplus \underbrace{\mathsf{H}^5_{\mathsf{dR}}(\mathsf{X}) \oplus \cdots}_{\mathsf{F}^{5/2}}}_{\mathsf{F}^{3/2}}}_{\mathsf{F}^{1/2}}$$

so in this case one can recover the individual de Rham cohomologies as the associated graded of the Hodge filtration.

For a smooth projective variety X, the noncommutative Hodge filtration on $\mathsf{HP}_\bullet(\mathsf{Perf}(\mathsf{X}))$ corresponds to the columns of the Hodge diamond $\mathsf{H}^{\bullet,\bullet} = \mathsf{H}^{\bullet,\bullet}(\mathsf{X})$:

$$\begin{array}{l}
\cdots \to \mathsf{F}^{3/2} \hookrightarrow \mathsf{F}^{1/2} \hookrightarrow \mathsf{F}^{-1/2} \hookrightarrow \mathsf{F}^{-3/2} \hookrightarrow \cdots \\
\cdots \hookrightarrow \mathsf{F}^{2/2} \hookrightarrow \mathsf{F}^{0} \hookrightarrow \mathsf{F}^{-2/2} \hookrightarrow \cdots
\end{array}$$

$$\begin{array}{ccccccc}
 & & & \mathsf{H}^{n,n} & & & \\
 & & \mathsf{H}^{n,n-1} & & \mathsf{H}^{n-1,n} & & \\
 & \mathsf{H}^{n,n-2} & & \mathsf{H}^{n-1,n-1} & & \mathsf{H}^{n-2,n} & \\
\mathsf{H}^{n,n-3} & & \mathsf{H}^{n-1,n-2} & & \mathsf{H}^{n-2,n-1} & & \mathsf{H}^{n-3,n} \\
\iddots & & & & & & \ddots \\
\ddots & & & & & & \iddots \\
\mathsf{H}^{3,0} & & \mathsf{H}^{2,1} & & \mathsf{H}^{1,2} & & \mathsf{H}^{0,3} \\
 & \mathsf{H}^{2,0} & & \mathsf{H}^{1,1} & & \mathsf{H}^{0,2} & \\
 & & \mathsf{H}^{1,0} & & \mathsf{H}^{0,1} & & \\
 & & & \mathsf{H}^{0,0} & & &
\end{array}$$

Here $\mathsf{HP}_{\mathsf{ev}}(\mathsf{Perf}(\mathsf{X}))$ is shown in black, while $\mathsf{HP}_{\mathsf{odd}}(\mathsf{Perf}(\mathsf{X}))$ is shown in blue. The symbol F^k is placed above the rightmost column that is included in the filtration component $\mathsf{F}^k\mathsf{HP}_\bullet(\mathsf{Perf}(\mathsf{X}))$. Everything to the left of that column and of the same colour is also included in that filtration component. Note that this filtration is different from the classical Hodge filtration, where the filtration is by the diagonals of the Hodge diamond, rather than the columns. Moreover the associated graded no longer recovers the individual Rham cohomologies, as those correspond to the rows, not the columns.

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