Noncommutative Hodge Theory
Lecture 3: The Hochschild–Kostant–Rosenberg theorem

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Abstract

We discuss the Hochschild–Kostant–Rosenberg theorem, which relates Hochschild homology to differential forms.

1 Overview

In the previous lecture, we introduced a natural invariant of an associative algebra $\mathcal{A}$, namely its Hochschild homology $\mathsf{HH}_\bullet(\mathcal{A})$. More generally, we associated to any dg category $\mathcal{C}$, its Hochschild homology $\mathsf{HH}_\bullet(\mathcal{C})$.

The goal for today is to give a geometric interpretation for Hochschild homology in the case of smooth algebraic varieties via the so-called Hochschild–Kostant–Rosenberg (HKR) theorem:

Theorem 1.1. *Let X be a smooth affine variety over an algebraically closed field $\mathbb{K}$ of characteristic zero, and let $\mathcal{A} = \mathcal{O}(\mathsf{X})$ be its algebra of functions. Then for every $k \geq 0$, we have a canonical isomorphism of $\mathcal{A}$-modules*

$$\mathsf{HH}_k(\mathcal{A}) \cong \Omega^k(\mathsf{X}) = \mathsf{H}^0(\mathsf{X}, \Omega^k_{\mathsf{X}})$$

where $\Omega^k(\mathsf{X})$ is the space of global algebraic k-forms on X.

This statement is for affine varieties, so it should be thought of as a local statement. There is also a global version that can be obtained by combining the HKR theorem above with results of Geller–Weibel, Keller, Loday and Thomason–Trobaugh:

Theorem 1.2. *Let X be a smooth, separated, quasi-compact scheme over $\mathbb{K}$. Then for any $k \in \mathbb{Z}$ there is a canonical isomorphism*

$$\mathsf{HH}_k(\mathsf{Perf}(\mathsf{X})) \cong \bigoplus_p \mathsf{H}^{p-k}(\mathsf{X}, \Omega^p_{\mathsf{X}}) = \bigoplus_p \mathsf{H}^{p,p-k}(\mathsf{X})$$

Here $\mathsf{Perf}(\mathsf{X})$ is the dg category of perfect complexes on X (complexes of quasi-coherent sheaves that are locally isomorphic to a bounded complex of finite-rank vector bundles). Notice that the Hochschild homology has only a single grading, whereas the groups $\mathsf{H}^{\bullet,\bullet}(\mathsf{X})$ have a bigrading. Indeed Hochschild homology corresponds precisely to the columns of the Hodge diamond:

$$
\begin{array}{ccccccc}
\cdots & \mathsf{HH}_2(\mathsf{X}) & \mathsf{HH}_1(\mathsf{X}) & \mathsf{HH}_0(\mathsf{X}) & \mathsf{HH}_{-1}(\mathsf{X}) & \mathsf{HH}_{-2}(\mathsf{X}) & \cdots \\
 & & & & & & \\
 & & & \mathsf{H}^{n,n}(\mathsf{X}) & & & \\
 & & \mathsf{H}^{n,n-1}(\mathsf{X}) & & \mathsf{H}^{n-1,n}(\mathsf{X}) & & \\
 & \mathsf{H}^{n,n-2}(\mathsf{X}) & & \mathsf{H}^{n-1,n-1}(\mathsf{X}) & & \mathsf{H}^{n-2,n}(\mathsf{X}) & \\
\iddots & & & & & & \ddots \\
\ddots & & & & & & \iddots \\
 & \mathsf{H}^{2,0}(\mathsf{X}) & & \mathsf{H}^{1,1}(\mathsf{X}) & & \mathsf{H}^{0,2}(\mathsf{X}) & \\
 & & \mathsf{H}^{1,0}(\mathsf{X}) & & \mathsf{H}^{0,1}(\mathsf{X}) & & \\
 & & & \mathsf{H}^{0,0}(\mathsf{X}) & & &
\end{array}
$$

Our aim in this lecture is to give some of the ideas that go into the proof of this fact.

2 HKR for affine varieties

2.1 Hochschild homology for the algebra of functions

We first consider the local case of an affine variety $\mathsf{X} = \mathsf{Spec}(\mathcal{A})$, where $\mathcal{A}$ is a commutative algebra over $\mathbb{K}$. We recall the following

Definition 2.1. The ***module of Kähler differentials*** $\Omega^1_{\mathcal{A}}$ is the $\mathcal{A}$-module characterized by the existence of a natural isomorphism

$$\mathsf{Hom}_{\mathcal{A}}(\Omega^1_{\mathcal{A}}, \mathcal{M}) \cong \mathcal{D}er_{\mathcal{A}}(\mathcal{M})$$

for all $\mathcal{A}$-modles $\mathcal{M}$. Here $\mathcal{D}er_{\mathcal{A}}(\mathcal{M})$ is the space of derivations from $\mathcal{A}$ to $\mathcal{M}$.

The Kähler differentials also have more concrete realizations. For instance, let $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ be the multiplication map, and let $\mathcal{I} = \ker \mu$. Then $\Omega^1_{\mathcal{A}} \cong \mathcal{I}/\mathcal{I}^2$. Alternatively, we have a generators-and-relations presentation of $\Omega^1_{\mathcal{A}}$: it is the $\mathcal{A}$-module generated by the formal symbols $a\,\mathrm{d}b$ for $a, b \in \mathcal{A}$, modulo the relations

$$\mathrm{d}(a+b) = \mathrm{d}a + \mathrm{d}b \qquad \mathrm{d}(ab) = a\mathrm{d}b + b\mathrm{d}a \qquad \mathrm{d}\lambda = 0$$

for all $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{K}$.

We claim that there is yet another description, namely $\Omega^1_{\mathcal{A}} = \mathsf{HH}_1(\mathcal{A})$. To see this, consider the relevant part of the Hochschild complex:

$$\begin{array}{ccccc} \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} & \xrightarrow{\mathrm{b}} & \mathcal{A} \otimes \mathcal{A} & \xrightarrow{\mathrm{b}} & \mathcal{A} \\ & & f \otimes g & \longmapsto & fg - gf = [f, g] \\ f \otimes g \otimes h & \longmapsto & fg \otimes h - f \otimes gh + hf \otimes g & & \end{array}$$

Since $\mathcal{A}$ is commutative, we see that every element in $\mathcal{A} \otimes \mathcal{A}$ is a cocycle, whereas the coboundaries can be written in the form

$$\mathrm{b}(f \otimes g \otimes h) = (f \otimes 1) \cdot (g \otimes h + h \otimes g - 1 \otimes gh)$$

From here, it follows easily that the map

$$\begin{array}{ccc} \mathcal{A} \otimes \mathcal{A} & \longrightarrow & \Omega^1_{\mathcal{A}} \\ f \otimes g & \longmapsto & f \,\mathrm{d}g \end{array}$$

descends to an isomorphism $\mathsf{HH}_1(\mathcal{A}) \cong \Omega^1_{\mathcal{A}}$.

More generally, we have a canonical map

$$\begin{array}{ccccc} \Psi & : & \mathsf{C}_k(\mathcal{A}) & \to & \Omega^k_{\mathcal{A}} \\ & & f_0 \otimes f_1 \otimes \cdots \otimes f_k & \mapsto & f_0 \,\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k \end{array} \tag{1}$$

that completely describes the Hochschild homology in the case of a smooth affine variety:

Theorem 2.2 (Hochschild–Kostant–Rosenberg [1])**.** *Suppose that $\mathcal{A}$ is a commutative algebra such that $\mathsf{X} = \mathsf{Spec}(\mathcal{A})$ is smooth. Then Ψ induces an isomorphism*

$$\mathsf{HH}_k(\mathcal{A}) \cong \Omega^k_{\mathcal{A}}$$

for all $k \in \mathbb{Z}$.

Remark 2.3. Notice that in this case, $\mathsf{HH}_\bullet(\mathcal{A})$ is an $\mathcal{A}$-algebra. This uses the commutativity of $\mathcal{A}$ in an essential way. □

Remark 2.4. This result justifies the viewpoint that if $\mathcal{A}$ is *any* algebra (possibly noncommutative), then we can think of $\mathsf{HH}_\bullet(\mathcal{A})$ as the space of differential forms on $\mathsf{Spec}(\mathcal{A})$. But so far we have not seen an important additional piece of structure: the de Rham differential. This will be covered next week. □

Let us recall that if $\mathcal{A}$ is a commutative algebra, then the smoothness of the scheme $\mathsf{X} = \mathsf{Spec}(\mathcal{A})$ corresponds to an algebraic condition on $\mathcal{A}$:

Definition 2.5. An algebra $\mathcal{A}$ is ***smooth*** if any of the following equivalent conditions hold:

1. The Hochschild homology is bounded, i.e. $\mathsf{HH}_k(\mathcal{A}) = 0$ for $k \gg 0$.

2. The module of Kähler forms $\Omega^1_{\mathcal{A}}$ is projective, i.e. the cotangent sheaf of $\mathsf{X} = \mathsf{Spec}(\mathcal{A})$ is a vector bundle.

3. For any maximal ideal $\mathfrak{m} \subset \mathcal{A}$, the kernel $\mathcal{I}_{\mathfrak{m}} = \ker(\mathcal{A}_{\mathfrak{m}} \otimes \mathcal{A}_m \to \mathcal{A}_{\mathfrak{m}})$ is generated by a regular sequence, where $\mathcal{A}_{\mathfrak{m}}$ denotes the localization of $\mathcal{A}$ at $\mathfrak{m}$.

Note that the first condition is clearly necessary if we want to have an isomorphism between $\mathsf{HH}_\bullet(\mathcal{A})$ and differential forms; the HKR theorem implies that it is also sufficient.

Sketch of the proof of Theorem 2.2. The induced map $\Psi : \mathsf{HH}_\bullet(\mathcal{A}) \to \Omega^\bullet_{\mathcal{A}}$ is a map of $\mathcal{A}$-modules. Hence it gives an isomorphism if and only if its localization $\Psi_{\mathfrak{m}} : \mathsf{HH}_\bullet(\mathcal{A})_{\mathfrak{m}} \to (\Omega^\bullet_{\mathcal{A}})_{\mathfrak{m}}$ at every maximal ideal $\mathfrak{m} \subset \mathcal{A}$ is an isomorphism.

But we claim there is a canonical isomorphism

$$\mathsf{HH}_\bullet(\mathcal{A})_{\mathfrak{m}} = (\mathsf{Tor}^{\mathcal{A}^{\mathsf{e}}}_\bullet(\mathcal{A}, \mathcal{A}))_{\mathfrak{m}} \cong \mathsf{Tor}^{(\mathcal{A}_{\mathfrak{m}})^{\mathsf{e}}}_\bullet(\mathcal{A}_{\mathfrak{m}}, \mathcal{A}_{\mathfrak{m}}) = \mathsf{HH}_\bullet(\mathcal{A}_{\mathfrak{m}})$$

In brief, the reason is that $(\mathcal{A} \otimes_{\mathcal{A}^{\mathsf{e}}} \mathcal{A})_{\mathfrak{m}} \cong \mathcal{A}_{\mathfrak{m}} \otimes_{\mathcal{A}^{\mathsf{e}}_{\mathfrak{m}}} \mathcal{A}_{\mathfrak{m}}$ and localization is exact; hence it commutes with derived functors.

In this way, we reduce the problem to the case of a local smooth $\mathbb{K}$-algebra $\mathcal{A}_{\mathfrak{m}}$. Then since $\mathfrak{m}$ is generated by a regular sequence, we can compute the Hochschild homology using the corresponding Koszul complex as we did for the polynomial ring in the previous lecture. This gives the desired result. □

2.2 Categorical version

Note that if $\mathcal{A}$ is an algebra, then instead of computing the Hochschild homology of $\mathcal{A}$, we could compute the Hochschild homology of the category $\mathsf{Perf}(\mathcal{A})$ of perfect $\mathcal{A}$-modules. A priori these seem like different constructions, but it turns out that they give the same answer.

Indeed, recall that we can view $\mathcal{A}$ as a category with one object $*$ whose endomorphism algebra is $\mathcal{A}$. Then we have a canonical functor

$$\begin{aligned} \mathcal{A} &\to \mathsf{Perf}(\mathcal{A}) \\ \{*\} &\mapsto \mathcal{A} \end{aligned}$$

where on the right we view $\mathcal{A}$ as a complex of $\mathcal{A}$-modules concentrated in degree zero.

Note that the category $\mathsf{Perf}(\mathcal{A})$ has many more objects than just $\mathcal{A}$, but in a sense they are all generated by $\mathcal{A}$; a perfect complex is, roughly speaking, one that is built out of free modules of finite rank. As a result, their Hochschild homologies are the same:

Theorem 2.6 (Keller [3]). *The functor $\mathcal{A} \to \mathsf{Perf}(\mathcal{A})$ induces an isomorphism*

$$\mathsf{HH}_\bullet(\mathcal{A}) \cong \mathsf{HH}_\bullet(\mathsf{Perf}(\mathcal{A}))$$

Applied to the category of perfect complexes on a smooth affine variety, we obtain the following

Corollary 2.7. *If* X *is a smooth affine variety, then we have canonical isomorphisms*

$$\mathsf{HH}_k(\mathcal{O}(\mathsf{X})) \cong \mathsf{HH}_k(\mathsf{Perf}(\mathsf{X})) \cong \Omega^k(\mathsf{X})$$

3 HKR for non-affine varieties

If X is a variety that is not necessarily affine, then there are a number of approaches one could use to define its Hochschild homology.

For instance, for any open set $\mathsf{U} \subset \mathsf{X}$, we have the algebra $\mathcal{O}_\mathsf{X}(\mathsf{U})$, and we can form its Hochschild complex $\mathsf{C}_\bullet(\mathcal{O}_\mathsf{X}(\mathsf{U}))$. This gives a presheaf of complexes on X, and we can form the sheafification, which we denote by $\mathsf{C}_\bullet(\mathcal{O}_\mathsf{X})$. Loday [4] then defines the Hochschild homology of X to be the hypercohomology of this sheaf of complexes:

$$\mathsf{HH}_k(\mathsf{X}) := \mathsf{H}^{-k}(\mathsf{X}, \mathsf{C}_\bullet(\mathcal{O}_\mathsf{X}))$$

Note that some care is needed here. Firstly, $\mathsf{HH}_\bullet(-)$ has a homological grading, but when we talk about hypercohomology we use a cohomological grading; as usual, this is done by putting $\mathsf{C}_k(\mathcal{O}_\mathsf{X})$ in cohomological degree $-k$. Secondly, and more importantly, we have to be careful defining the hypercohomology, since the complex of sheaves $\mathsf{C}_\bullet(\mathcal{O}_\mathsf{X})$ is not bounded below. However, once things are correctly defined, this definition is consistent with the previous definition for affine varieties:

Theorem 3.1 (Geller–Weibel [6, Section 4])**.** *Let* X *be an affine scheme. Then Loday's sheaf-theoretic definition of Hochschild homology agrees with the usual Hochschild homology of the coordinate ring:*

$$\mathsf{H}^\bullet(\mathsf{X}, \mathsf{C}_\bullet(\mathcal{O}_\mathsf{X})) \cong \mathsf{HH}_\bullet(\mathcal{O}(\mathsf{X}))$$

In other words, Hochschild homology satisfies descent with respect to Zariski covers.

Evidently, the HKR map (1) can be sheafified to give a morphism of complexes

$$\mathsf{C}_\bullet(\mathsf{X}, \mathcal{O}_\mathsf{X}) \to \left(\cdots \longrightarrow \Omega^2_\mathsf{X} \xrightarrow{0} \Omega^1_\mathsf{X} \xrightarrow{0} \mathcal{O}_\mathsf{X} \right)$$

and if X is smooth, the HKR theorem implies that this is a quasi-isomorphism of complexes of sheaves. Since the differential on the right hand side is trivial, the hypercohomology reduces to the ordinary cohomology, and keeping track of the degrees, we immediately obtain the following result

Proposition 3.2. *If* X *is a smooth variety and we use Loday's sheaf-theoretic definition of Hochschild homology, then we have the global Hochschild–Kostant–Rosenberg isomorphism*

$$\mathsf{HH}_k(\mathsf{X}) \cong \bigoplus_p \mathsf{H}^{p-k}(\mathsf{X}, \Omega^p_\mathsf{X}) = \bigoplus_p \mathsf{H}^{p,p-k}(\mathsf{X})$$

A second approach to defining the Hochschild homology of a non-affine variety X is to recall that X is determined by its category of sheaves. Thus, for instance, we could consider the Hochschild homology of the dg category $\mathsf{Perf}(\mathsf{X})$ of perfect complexes on X. But this gives the same answer as Loday's sheaf-theoretic construction:

Theorem 3.3 (Keller [2, Section 5.2])**.** *If X is a quasi-compact separated scheme, (e.g. if X is quasi-projective) then there is an isomorphism*

$$\mathsf{HH}_\bullet(\mathsf{Perf}(\mathsf{X})) \cong \mathsf{HH}_\bullet(\mathsf{X})$$

To prove this result, Keller constructs a complex of sheaves $\mathsf{C}_\bullet(\mathsf{Perf}(\mathcal{O}_\mathsf{X}))$ by sheafifying the presheaf $\mathsf{U} \mapsto \mathsf{C}_\bullet(\mathsf{Perf}(\mathcal{O}_\mathsf{X}(\mathsf{U})))$ of categorical Hochschild chain complexes. Using the compatibility between Hochschild homologies of algebras and categories (Theorem 2.6) he proves that this complex is equivalent to Loday's: $\mathsf{C}_\bullet(\mathsf{Perf}(\mathcal{O}_\mathsf{X})) \cong \mathsf{C}_\bullet(\mathcal{O}_\mathsf{X})$. Because $\mathsf{C}_\bullet(\mathsf{Perf}(\mathcal{O}_\mathsf{X}))$ was constructed by sheafification, there is a natural map

$$\tau : \mathsf{HH}_\bullet(\mathsf{Perf}(\mathsf{X})) \to \mathsf{H}^\bullet(\mathsf{X}, \mathsf{C}_\bullet(\mathsf{Perf}(\mathcal{O}_\mathsf{X}))) \cong \mathsf{HH}_\bullet(\mathsf{X}).$$

That τ is an isomorphism when X is affine is essentially Theorem 2.6 and Theorem 3.1. But an arbitrary quasi-compact scheme has a finite cover by affines, so to prove that τ is an isomorphism for general X, it is enough to control what happens when finitely many affines are glued together. To this end, Keller proves that an analogue of the Meyer–Vietoris sequence holds; this, in turn, uses a theorem of Thomason–Trobaugh [5], which explains how categories of sheaves localize on Zariski open sets.

Putting all these results together, we arrive at the categorical version of the HKR theorem: for a smooth quasi-projective variety X, there is a canonical isomorphism

$$\mathsf{HH}_k(\mathsf{Perf}(\mathsf{X})) \cong \bigoplus_p \mathsf{H}^{p,p-k}(\mathsf{X})$$

for all $k \in \mathbb{Z}$.

References

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