

Noncommutative Hodge Theory

Lecture 2: Hochschild Homology

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Abstract

We introduce one of the basic ingredients of noncommutative Hodge theory: Hochschild homology of algebras and categories.

1 Geometric motivation

Differential forms on a variety X and their sheaf cohomology groups

$$H^{p,q}(X) = H^q(X, \Omega_X^p)$$

play an essential role in defining the Hodge structure on the cohomology of X . Thus the first step towards defining noncommutative Hodge structures is to find a good replacement for the groups $H^{p,q}(X)$ when X is a noncommutative space.

In this lecture, we introduce an important invariant of algebras and categories: the Hochschild homology $HH_\bullet(-)$. Standard references for this material include [3, 6]. In the next lecture we will see that, when applied to the algebra of functions (or category of sheaves) on a smooth variety X , the Hochschild homology recovers the spaces $H^{p,q}(X)$.

2 Hochschild homology for algebras

Let \mathbb{K} be a field. Throughout these notes, all algebras are unital \mathbb{K} -algebras, i.e. they contain a multiplicative identity 1. The tensor product symbol \otimes denotes the tensor product over \mathbb{K} unless otherwise specified.

Definition 2.1. Let \mathcal{A} be a \mathbb{K} -algebra. The *Hochschild complex of \mathcal{A}* is

$$C_\bullet(\mathcal{A}) = \left(\cdots \xrightarrow{b} \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \xrightarrow{b} \mathcal{A} \otimes \mathcal{A} \xrightarrow{b} \mathcal{A} \right)$$

where \mathcal{A} sits in degree zero, and the differential is given by

$$b(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^n a_n a_0 \otimes \cdots \otimes a_{n-1}$$

for all $a_0, \dots, a_n \in \mathcal{A}$. The **Hochschild homology of \mathcal{A}** is the homology of this complex: $\mathrm{HH}_\bullet(\mathcal{A}) = \mathrm{H}_\bullet(\mathrm{C}_\bullet(\mathcal{A}), \mathrm{b})$.

Let us compute the differential b in the simplest case. Given $a_0, a_1 \in \mathcal{A}$ we have that

$$\mathrm{b}(a_0 \otimes a_1) = a_0 a_1 - a_1 a_0 = [a_0, a_1],$$

the commutator of a_0 and a_1 . Thus the zeroth Hochschild homology has a simple interpretation, as the **cocentre**:

$$\mathrm{HH}_0(\mathcal{A}) = \frac{\mathrm{C}_0(\mathcal{A})}{\mathrm{d}(\mathrm{C}_1(\mathcal{A}))} = \frac{\mathcal{A}}{\mathrm{d}(\mathcal{A} \otimes \mathcal{A})} = \frac{\mathcal{A}}{[\mathcal{A}, \mathcal{A}]}$$

where $[\mathcal{A}, \mathcal{A}] \subset \mathcal{A}$ is the subspace spanned by all commutators of \mathcal{A} .

In general, the homology of the Hochschild complex is quite difficult to compute directly. For this reason, it is helpful to give an interpretation of $\mathrm{HH}_\bullet(\mathcal{A})$ as a derived functor, which often allows us to compute it using a simpler complex.

To this end, we recall that the **enveloping algebra of \mathcal{A}** is the algebra

$$\mathcal{A}^e := \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}},$$

where $\mathcal{A}^{\mathrm{op}}$ denotes the opposite algebra of \mathcal{A} . By construction, a left (or right) \mathcal{A}^e -module is the same thing as an \mathcal{A} -bimodule. In particular, \mathcal{A} is both a left and a right \mathcal{A}^e -module, so that we may define the tensor product $\mathcal{A} \otimes_{\mathcal{A}^e} \mathcal{A}$, and its derived version, the Tor groups $\mathrm{Tor}_\bullet^{\mathcal{A}^e}(\mathcal{A}, \mathcal{A})$.

Proposition 2.2. *We have a canonical isomorphism*

$$\mathrm{HH}_\bullet(\mathcal{A}) \cong \mathrm{Tor}_\bullet^{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}).$$

Sketch of proof. (See, e.g. [3, Section 1.1] for more details.) To compute the Tor group, we find a resolution of \mathcal{A} by free \mathcal{A}^e -modules. This resolution is given by the bar complex

$$\mathrm{Bar}_\bullet(\mathcal{A}) = \left(\cdots \xrightarrow{\mathrm{b}'} \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \xrightarrow{\mathrm{b}'} \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \xrightarrow{\mathrm{b}'} \mathcal{A} \otimes \mathcal{A} \right)$$

where the differential is given by

$$\mathrm{b}'(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$$

Note that this is similar to the Hochschild differential, but with less terms. If we view $\mathcal{A}^{\otimes n}$ as an \mathcal{A} -bimodule by multiplication on the leftmost and rightmost tensor factors, then $\mathrm{Bar}_\bullet(\mathcal{A})$ becomes a complex of free \mathcal{A}^e -modules.

Since \mathcal{A} is unital, multiplication gives a natural surjective \mathcal{A} -bimodule map

$$\mathrm{Bar}_0(\mathcal{A}) = \mathcal{A} \otimes \mathcal{A} \twoheadrightarrow \mathcal{A}$$

and one can show that this induces a quasi-isomorphism $\mathrm{Bar}_\bullet(\mathcal{A}) \cong \mathcal{A}$ of complexes of \mathcal{A} -bimodules. Hence we may compute $\mathrm{Tor}_\bullet^{\mathcal{A}^e}(\mathcal{A}, \mathcal{A})$ as the homology of the complex $\mathrm{Bar}_\bullet(\mathcal{A}) \otimes_{\mathcal{A}^e} \mathcal{A}$, but this is canonically isomorphic to the Hochschild complex $\mathrm{C}_\bullet(\mathcal{A})$, which gives the result. \square

Remark 2.3. When \mathcal{A} is commutative, this proposition has a geometric interpretation in terms of the scheme $\mathbf{X} = \mathrm{Spec}(\mathcal{A})$. Indeed $\mathcal{A}^e = \mathcal{O}(\mathbf{X}) \otimes \mathcal{O}(\mathbf{X})$ can be viewed as the algebra $\mathcal{O}(\mathbf{X} \times \mathbf{X})$ of functions on the product $\mathbf{X} \times \mathbf{X}$. The multiplication map $\mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ used in the bar resolution is dual to the diagonal inclusion $\mathbf{X} \hookrightarrow \mathbf{X} \times \mathbf{X}$. The tensor product $\mathcal{A} \otimes_{\mathcal{A}^e} \mathcal{A} \cong \mathcal{A}$ is therefore naturally interpreted as the algebra of functions on the self-intersection $\mathbf{X} \cap \mathbf{X} = \mathbf{X}$ of the diagonal inside $\mathbf{X} \times \mathbf{X}$. This intersection is not transverse, and the failure of transversality is measured by the higher Tor groups $\mathrm{Tor}_\bullet^{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}) = \mathrm{HH}_\bullet(\mathcal{A})$. Indeed, one of the key ideas of derived algebraic geometry (originating from an intersection multiplicity formula due to Serre) is that the higher Tor groups should be considered part of the *definition* of $\mathbf{X} \cap \mathbf{X}$ as a “derived scheme”. \square

Using [Proposition 2.2](#), we can determine the Hochschild homology of some simple algebras.

Example 2.4. Let $\mathcal{A} = \mathbb{K}[x_1, \dots, x_n]$ be a polynomial ring. Then the enveloping algebra is also a polynomial ring, whose generators we give different names.

$$\mathcal{A}^e = \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \cong \mathbb{K}[y_1, \dots, y_n, z_1, \dots, z_n]$$

Under this identification, the multiplication map $\mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ is given by the ring homomorphism that sends $y_i \mapsto x_i$ and $z_i \mapsto x_i$. Evidently the elements $r_i = y_i - z_i$ are annihilated by this map, and in fact they generate the kernel, so that we have a presentation

$$\mathcal{A} = \mathcal{A}^e / (r_1, \dots, r_n)$$

In other words, if $\mathcal{M} := (\mathcal{A}^e)^{\oplus n}$, we have a map

$$\mathcal{M} \xrightarrow{(r_1, \dots, r_n)} \mathcal{A}^e$$

whose cokernel is isomorphic to \mathcal{A} . This extends to the Koszul complex

$$(\wedge^\bullet \mathcal{M}, d) = \left(\dots \longrightarrow \wedge^3 \mathcal{M} \longrightarrow \wedge^2 \mathcal{M} \longrightarrow \mathcal{M} \longrightarrow \mathcal{A}^e \right)$$

where $\wedge^k \mathcal{M}$ denotes the k th exterior power of \mathcal{M} as an \mathcal{A}^e -module, and where the differential is given in terms of the basis elements $e_i \in \mathcal{M}$ by the formula

$$d(e_{i_1} \wedge \dots \wedge e_{i_p}) = \sum_{k=1}^p (-1)^k r_{i_k} e_{i_1} \wedge \dots \wedge \widehat{e_{i_k}} \wedge \dots \wedge e_{i_p}.$$

One can show that the Koszul complex gives a free resolution of \mathcal{A} as an \mathcal{A}^e -module; this uses the fact that the elements r_1, \dots, r_n form a regular sequence (see, e.g. [1, Chapter 17]).

If we let $\mathcal{N} = \mathcal{M} \otimes_{\mathcal{A}^e} \mathcal{A} \cong \mathcal{A}^{\oplus n}$ then by [Proposition 2.2](#), the Hochschild homology of \mathcal{A} is the homology of the complex $(\wedge^\bullet \mathcal{M}, d) \otimes_{\mathcal{A}^e} \mathcal{A} \cong (\wedge^\bullet \mathcal{N}, d = 0)$. Therefore

$$\mathrm{HH}_\bullet(\mathcal{A}) \cong \wedge^\bullet \mathcal{N}$$

where \mathcal{N} is a free \mathcal{A} -module of rank n . In the next lecture, we will see that \mathcal{N} is best interpreted as the differential one-forms on the affine space $\mathbb{A}^n = \mathrm{Spec}(\mathcal{A})$, and view the result above as a special case of the Hochschild–Kostant–Rosenberg theorem. \square

Example 2.5. Let \mathfrak{g} be a finite-dimensional Lie algebra over \mathbb{K} , and let $\mathcal{U}(\mathfrak{g})$ be its universal enveloping algebra. Thus $\mathcal{U}(\mathfrak{g})$ is the quotient

$$\mathcal{U}(\mathfrak{g}) = \frac{\mathrm{T}(\mathfrak{g})}{(x \otimes y - y \otimes x - [x, y])}$$

where $\mathrm{T}(\mathfrak{g}) = \mathbb{K} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus \cdots$ is the tensor algebra of \mathfrak{g} . By construction, a left $\mathcal{U}(\mathfrak{g})$ -module is the same data as a representation of the Lie algebra \mathfrak{g} .

Let \mathfrak{g}_t denote the Lie algebra whose bracket is given by rescaling the bracket on \mathfrak{g} by the constant t , i.e. $[-, -]_t = t[-, -]$ for $t \in \mathbb{K}$. When $t = 0$, the Lie algebra \mathfrak{g}_t is abelian and its universal enveloping algebra is simply the symmetric algebra $\mathcal{U}(\mathfrak{g}_t) = \mathrm{Sym}(\mathfrak{g})$, i.e. a polynomial ring. We can view it as the algebra of functions on the affine space \mathfrak{g}^\vee (the dual vector space of \mathfrak{g}). If $[-, -] \neq 0$, then $\mathcal{U}(\mathfrak{g}_t)$ is noncommutative for $t \neq 0$, but the Poincaré–Birkhoff–Witt theorem states that there is a natural isomorphism of vector spaces $\mathrm{Sym}(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{g})$, given by the symmetrization of the product on $\mathcal{U}(\mathfrak{g})$. Thus we can view the product on $\mathcal{U}(\mathfrak{g})$ as a noncommutative deformation (quantization) of the product on the commutative ring $\mathrm{Sym}(\mathfrak{g})$.

Correspondingly, the Hochschild complex of $\mathcal{U}(\mathfrak{g})$ can be computed using a deformation of the Koszul complex from [Example 2.4](#). More precisely, we have a complex $(\mathcal{U}(\mathfrak{g}) \otimes \wedge^\bullet \mathfrak{g}, d)$, where the differential is given by

$$\begin{aligned} d(u \otimes x_1 \wedge \cdots \wedge x_n) &= \sum_{i=1}^n (-1)^i (x_i u - u x_i) \otimes x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_n \\ &\quad + \sum_{i < j} (-1)^{i+j} u \otimes [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_n \end{aligned}$$

This is the complex that computes the Chevalley–Eilenberg Lie algebra homology of $\mathcal{U}(\mathfrak{g})$ as a right \mathfrak{g} -module, where the action of $x \in \mathfrak{g}$ on $u \in \mathcal{U}(\mathfrak{g})$ is given by $u \cdot x = ux - xu$ for $x \in \mathfrak{g}$ and $u \in \mathcal{U}(\mathfrak{g})$; in other words, \mathfrak{g} is acting via the adjoint representation on the bimodule $\mathcal{U}(\mathfrak{g})$.

One can check that the antisymmetrization map

$$\begin{aligned} \mathcal{U}(\mathfrak{g}) \otimes \wedge^\bullet \mathfrak{g} &\rightarrow \mathbf{C}_\bullet(\mathcal{U}(\mathfrak{g})) \\ u \otimes g_1 \wedge \cdots \wedge g_p &\mapsto \sum_{\sigma \in S_p} \mathrm{sign}(\sigma) u \otimes g_{\sigma_1} \otimes \cdots \otimes g_{\sigma_p} \end{aligned}$$

is a quasi-isomorphism (see [3, Section 3.3.1] for details), so that we have an isomorphism

$$\mathrm{HH}_\bullet(\mathcal{U}(\mathfrak{g})) \cong \mathrm{H}_\bullet(\mathfrak{g}, \mathcal{U}(\mathfrak{g}))$$

between the Hochschild homology and the Lie algebra homology. \square

3 Hochschild homology of dg algebras

We now briefly explain how to extend the definition of Hochschild homology to the more general setting of differential graded algebras.

Definition 3.1. A (unital) *differential graded (dg) algebra* is a cochain complex (\mathcal{A}, δ) with a maps of complexes

$$\mu : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$$

that gives an associative product on \mathcal{A} , and a cocycle $1 \in \mathcal{A}$ that is the multiplicative unit.

A dg algebra is, in particular, an algebra, so we can form the usual Hochschild differential

$$\dots \xrightarrow{\mathrm{b}} \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \xrightarrow{\mathrm{b}} \mathcal{A} \otimes \mathcal{A} \xrightarrow{\mathrm{b}} \mathcal{A}$$

but now since \mathcal{A} is itself a complex, this is really a double complex, where the vertical arrows are the usual differentials on the tensor product of complexes:

$$\begin{array}{ccccc} & & \vdots & & \vdots & & \vdots \\ & & \uparrow \delta & & \uparrow \delta & & \uparrow \delta \\ \dots & \xrightarrow{\mathrm{b}} & (\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A})^1 & \xrightarrow{\mathrm{b}} & (\mathcal{A} \otimes \mathcal{A})^1 & \xrightarrow{\mathrm{b}} & \mathcal{A}^1 \\ & & \uparrow \delta & & \uparrow \delta & & \uparrow \delta \\ \dots & \xrightarrow{\mathrm{b}} & (\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A})^0 & \xrightarrow{\mathrm{b}} & (\mathcal{A} \otimes \mathcal{A})^0 & \xrightarrow{\mathrm{b}} & \mathcal{A}^0 \\ & & \uparrow \delta & & \uparrow \delta & & \uparrow \delta \\ \dots & \xrightarrow{\mathrm{b}} & (\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A})^{-1} & \xrightarrow{\mathrm{b}} & (\mathcal{A} \otimes \mathcal{A})^{-1} & \xrightarrow{\mathrm{b}} & \mathcal{A}^{-1} \\ & & \uparrow \delta & & \uparrow \delta & & \uparrow \delta \\ & & \vdots & & \vdots & & \vdots \end{array}$$

The *Hochschild complex of \mathcal{A}* is then the direct sum total complex of this double complex:

$$\mathrm{C}_\bullet(\mathcal{A}) = \mathrm{Tot}^\oplus \left(\dots \xrightarrow{\mathrm{b}} \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \xrightarrow{\mathrm{b}} \mathcal{A} \otimes \mathcal{A} \xrightarrow{\mathrm{b}} \mathcal{A} \right)$$

If \mathcal{A} is an ordinary algebra, viewed as a dg algebra concentrated in degree zero, this clearly recovers the previous definition.

4 Differential graded categories

Finally, we extend to the setting of differential graded categories, which is where many of the most interesting examples are situated. Standard references include [2, 5].

Definition 4.1. A \mathbb{K} -linear *differential graded category* (*dg category* for short) is a category \mathcal{C} enriched in complexes of \mathbb{K} -vector spaces. More concretely, it consists of

- a collection of objects $\mathcal{M}, \mathcal{N}, \dots \in \mathcal{C}$.
- a complex $\mathcal{C}(\mathcal{M}, \mathcal{N}) = \text{Hom}_{\mathcal{C}}^{\bullet}(\mathcal{M}, \mathcal{N})$ of morphisms for every pair of objects x, y
- composition maps

$$\mathcal{C}(\mathcal{M}, \mathcal{N}) \otimes \mathcal{C}(\mathcal{L}, \mathcal{M}) \rightarrow \mathcal{C}(\mathcal{L}, \mathcal{N})$$

that are morphisms of complexes, and satisfy the usual associative law from compositions: $(f \circ g) \circ h = f \circ (g \circ h)$.

- a cocycle $1 \in \mathcal{C}(\mathcal{M}, \mathcal{M})$ for every object \mathcal{M} , that acts as the identity for the composition law

If \mathcal{C} is a dg category, then we obtain a \mathbb{K} -linear category $\mathbf{H}^0(\mathcal{C})$ by taking zeroth cohomology of all the morphism complexes. This is called the *homotopy category of \mathcal{C}* .

Example 4.2. Any \mathbb{K} -linear category is a dg category \mathcal{C} where the morphism complex is concentrated in degree zero. In this case $\mathbf{H}^0(\mathcal{C}) = \mathcal{C}$. \square

Example 4.3. A dg algebra \mathcal{A} can be viewed as a dg category with a single object $*$ $\in \mathcal{C}$ whose endomorphisms are given by $\mathcal{C}(*, *) = \mathcal{A}$. In this case $\mathbf{H}^0(\mathcal{C})$ is the category with one object $*$ and endomorphism algebra $\mathbf{H}^0(\mathcal{A})$. \square

Example 4.4. If \mathcal{A} is an algebra (or more generally, a dg algebra), then there is a dg category $\text{Cplx}(\mathcal{A})$ whose objects are complexes of \mathcal{A} -modules (resp. dg modules over \mathcal{A}). Give a pair of complexes $(\mathcal{M}, d_{\mathcal{M}})$ and $(\mathcal{N}, d_{\mathcal{N}})$, the complex of morphisms $\text{Hom}_{\text{Cplx}(\mathcal{A})}^{\bullet}(\mathcal{M}, \mathcal{N})$ has degree- n piece given by

$$\text{Hom}_{\text{Cplx}(\mathcal{A})}^n(\mathcal{M}, \mathcal{N}) = \prod_{i=-\infty}^{\infty} \text{Hom}_{\mathcal{A}}(\mathcal{M}^i, \mathcal{N}^{i+n})$$

the space of degree- n maps of graded \mathcal{A} -modules. The differential is given by $d f = d_{\mathcal{N}} f - (-1)^n f d_{\mathcal{M}}$ where $f \in \text{Hom}_{\text{Cplx}(\mathcal{A})}^n(\mathcal{M}, \mathcal{N})$. Note that the degree zero cocycles are precisely the cochain maps, and the degree-zero coboundaries are the null-homotopic cochain maps. Hence $\mathbf{H}^0(\text{Cplx}(\mathcal{A}))$ is the homotopy category of complexes of \mathcal{A} -modules. \square

Example 4.5. The category of **perfect complexes of \mathcal{A} -modules** is the full subcategory

$$\text{Perf}(\mathcal{A}) \subset \text{Cplx}(\mathcal{A})$$

consisting of complexes that are quasi-isomorphic to a bounded complex of projective \mathcal{A} -modules of finite rank. \square

Example 4.6. Let X be a quasi-compact quasi-separated scheme (we need this technical condition to ensure that $\text{QCoh}(X)$ is Grothendieck abelian, i.e. a category we can do homological algebra with). Then there are dg categories

$$\text{Perf}(X) \subset \text{Coh}_{\text{dg}}(X) \subset \text{QCoh}_{\text{dg}}(X)$$

whose homotopy categories are the bounded derived categories

$$D^b\text{Perf}(X) \subset D^b\text{Coh}(X) \subset D^b\text{QCoh}(X)$$

of perfect/coherent/quasicoherent sheaves on X . The construction of these dg categories is slightly subtle because $\text{QCoh}(X)$ does not have enough projectives. One approach is to use take the subcategory of the dg category of chain complexes in $\text{QCoh}(X)$, consisting of complexes that are injective as \mathcal{O}_X -modules and locally quasi-isomorphic to perfect/coherent/quasi-coherent complexes. \square

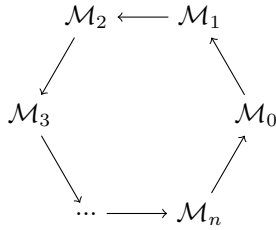
It is straightforward to extend the definition of Hochschild homology to arbitrary (small) dg categories (c.f. [2], Section 5.3). Indeed, suppose that \mathcal{C} is a dg category. Then we may define a double complex

$$\cdots \xrightarrow{b} \bigoplus_{\mathcal{M}_0, \mathcal{M}_1 \in \mathcal{C}} \mathcal{C}(\mathcal{M}_0, \mathcal{M}_1) \otimes \mathcal{C}(\mathcal{M}_1, \mathcal{M}_0) \xrightarrow{b} \bigoplus_{\mathcal{M}_0 \in \mathcal{C}} \mathcal{C}(\mathcal{M}_0, \mathcal{M}_0)$$

where the n th column is given by the space

$$\bigoplus_{\mathcal{M}_0, \mathcal{M}_1, \dots, \mathcal{M}_n \in \mathcal{C}} \mathcal{C}(\mathcal{M}_0, \mathcal{M}_1) \otimes \mathcal{C}(\mathcal{M}_1, \mathcal{M}_2) \otimes \cdots \otimes \mathcal{C}(\mathcal{M}_n, \mathcal{M}_0)$$

formed from morphisms in \mathcal{C} that compose in a cycle:



The horizontal differential is given by

$$b(f_0 \otimes \cdots \otimes f_n) = \sum_{i=0}^{n-1} \pm f_0 \otimes \cdots \otimes (f_{i+1} \circ f_i) \otimes \cdots \otimes f_n \pm f_0 \circ f_n \otimes \cdots \otimes f_{n-1}.$$

and the vertical differential is the usual one on the tensor product. See [4] for a careful uniform treatment of the signs in the even more general context of A_∞ categories.

Denoting by $C_\bullet(\mathcal{C})$ the direct sum total complex of this double complex, the **Hochschild homology** of \mathcal{C} is the cohomology $\mathrm{HH}_\bullet(\mathcal{C}) := H_\bullet(C_\bullet(\mathcal{C}), b)$. It's not hard to see that if \mathcal{C} is a dga, then this agrees with the Hochschild homology defined earlier.

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