Noncommutative Hodge Theory
Lecture 1: Introduction

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Classical Hodge theory is a package of results and techniques concerning the cohomology of algebraic varieties. Over the years, it has evolved into a tremendously powerful tool, with applications to the classification of algebraic varieties, the topology of singularities, the study of period integrals in transcendental number theory, and more.

Our aim in this learning seminar is discuss recent proposals and results that seek to extend Hodge theory to the much broader setting of "noncommutative algebraic varieties", i.e. certain noncommutative rings (or categories) that behave as though they were the coordinate rings (or categories of sheaves) on a classical algebraic variety. This ***noncommutative Hodge theory*** is expected to have interesting applications in areas such as mirror symmetry and deformation quantization, where classical geometry interacts with noncommutative and categorical structures in a complicated way.

We are trying to keep the prerequisites to a minimum, but will have to assume some previous knowledge of sheaf theory, de Rham cohomology, and homological algebra. We will spend most of the term simply introducing the various ingredients of the theory (certain cohomological invariants of noncommutative rings and categories) and examining their basic properties. In this lecture we start with a brief review of classical Hodge theory, so that we have some idea what we are trying to generalize.

1 Elliptic curves

To warm up, let us recall one of the basic facts of complex geometry: an elliptic curve is determined up to isomorphism by the structure of its cohomology.

This works as follows. Suppose that X is an elliptic curve, i.e. a one-dimensional complex manifold of the form

$$\mathsf{X} = \mathbb{C}/\Lambda$$

where $\Lambda \cong \mathbb{Z}^2 \subset \mathbb{C}$ is a lattice. Each such X is topologically a two-torus, but the complex structure depends on the lattice Λ.

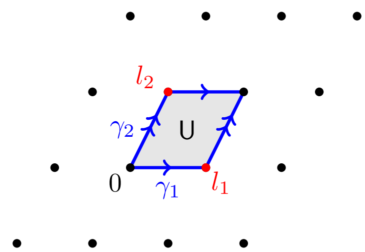


Figure 1: A lattice $\Lambda \subset \mathbb{C}$ with basis l_1, l_2 and fundamental domain U. The curves γ_1, γ_2 form a basis for the first homology of the elliptic curve $\mathsf{X} = \mathbb{C}/\Lambda$.

If z denotes the standard coordinate function on $\mathbb{C}$, then the one-form $\mathrm{d}z$ on $\mathbb{C}$ is invariant under translations, and hence it descends to a global holomorphic one-form α on X. We can recover the lattice $\Lambda \subset \mathbb{C}$, and hence X itself, by integrating α over cycles in X. More precisely, suppose that $l_1, l_2 \in \Lambda \subset \mathbb{C}$ are a basis for Λ as a $\mathbb{Z}$-module, and let $\gamma_1, \gamma_2 \in \mathsf{H}_\bullet(\mathsf{X}; \mathbb{Z})$ be the one-cycles corresponding to the straight line paths from the origin each basis vector. We can then recover l_1, l_2 as

$$l_i = \int_0^{l_i} \mathrm{d}z = \int_{\gamma_i} \alpha$$

and hence we get the lattice $\Lambda = \langle l_1, l_2 \rangle$

We can phrase this construction more invariantly (i.e. without choosing bases and coordinates) as follows. Since α non-vanishing, any other holomorphic one-form on X must be given by multiplying α by a holomorphic function. But X is compact, so such a function is necessarily constant, and we conclude that α actually gives a basis for the space of holomorphic forms:

$$\mathsf{H}^0(\mathsf{X}, \Omega^1_{\mathsf{X}}) = \mathbb{C} \cdot \{\alpha\}$$

Here $\mathsf{H}^0(\mathsf{X}, \Omega^1_{\mathsf{X}})$ denotes the zeroth cohomology (global sections) of the sheaf of holomorphic one-forms on X. The advantage of this description is that it is intrinsic to X as an abstract complex manifold, whereas the space on the right used a choice of coordinates and an explicit presentation of X as a quotient $\mathbb{C}/\Lambda$.

Since α is closed, it defines a class in the de Rham cohomology

$$[\alpha] \in \mathsf{H}^1_{\mathsf{dR}}(\mathsf{X})$$

The same can be said of its complex conjugate

$$[\overline{\alpha}] \in \mathsf{H}^1_{\mathsf{dR}}(\mathsf{X})$$

We claim that in fact $[\alpha]$ and $[\overline{\alpha}]$ give a basis for $\mathsf{H}^1_{\mathsf{dR}}(\mathsf{X})$. More invariantly, we have a canonical decomposition

$$\mathsf{H}^1_{\mathsf{dR}}(\mathsf{X}) \cong \mathsf{H}^0(\mathsf{X}, \Omega^1_{\mathsf{X}}) \oplus \overline{\mathsf{H}^0(\mathsf{X}, \Omega^1_{\mathsf{X}})} =: \mathsf{H}^{1,0}(\mathsf{X}) \oplus \mathsf{H}^{0,1}(\mathsf{X})$$

To see this, note that $\mathsf{H}^1_{\mathsf{dR}}(\mathsf{X})$ is two-dimensional, because X is a two-torus, so it is enough to check that α and $\overline{\alpha}$ are linearly independent. But $\alpha \wedge \overline{\alpha}$ corresponds to the two-form $\mathrm{d}z \wedge \mathrm{d}\overline{z} = -2\mathrm{i}\mathrm{d}x \wedge \mathrm{d}y$ on $\mathbb{C}$, where $z = x + \mathrm{i}y$. Therefore

$$\begin{aligned} \int_{\mathsf{X}} \alpha \wedge \overline{\alpha} &= -2\mathrm{i} \int_{\mathsf{U}} \mathrm{d}x \wedge \mathrm{d}y \\ &= -2\mathrm{i} \cdot (\text{area of } \mathsf{U}) \\ &\neq 0 \end{aligned}$$

where $\mathsf{U} \subset \mathbb{C}$ is a fundamental domain for the lattice $\Lambda \subset \mathbb{C}$ as shown in Figure 1. Therefore α and $\overline{\alpha}$ are linearly independent, as desired.

Finally, we observe that integration gives a pairing

$$\int : \mathsf{H}_1(\mathsf{X}; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathsf{H}^1_{\mathsf{dR}}(\mathsf{X}) \to \mathbb{C}$$

By de Rham's theorem this gives a perfect pairing of $\mathbb{C}$-vector spaces

$$\mathsf{H}_1(\mathsf{X}; \mathbb{C}) \otimes_{\mathbb{C}} \mathsf{H}^1_{\mathsf{dR}}(\mathsf{X}) \to \mathbb{C}$$

where

$$\mathsf{H}_1(\mathsf{X}; \mathbb{C}) := \mathsf{H}_1(\mathsf{X}; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}.$$

is the homology with complex coefficients. Put differently, if $\mathsf{H}^1(\mathsf{X}; \mathbb{Z})$ denotes the singular cohomology, then we have

$$\mathsf{H}^1(\mathsf{X}; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathsf{H}^1_{\mathsf{dR}}(\mathsf{X}).$$

Now by considering the composition

$$\mathbb{Z}^2 \cong \mathsf{H}^1(\mathsf{X}; \mathbb{Z}) \hookrightarrow \mathsf{H}^1(\mathsf{X}; \mathbb{C}) \xrightarrow{\sim} \mathsf{H}^1_{\mathsf{dR}}(\mathsf{X}) \twoheadrightarrow \mathsf{H}^{1,0}(\mathsf{X}) \cong \mathbb{C}$$

we get an abstract lattice $\mathsf{H}^1(\mathsf{X}; \mathbb{Z}) \hookrightarrow \mathsf{H}^{1,0}(\mathsf{X})$ isomorphic to the lattice $\Lambda \subset \mathbb{C}$ we started with. Hence the quotient $\mathsf{H}^{1,0}(\mathsf{X})/\mathsf{H}^1(\mathsf{X}; \mathbb{Z})$ is an elliptic curve that is isomorphic to X. But here we have only used constructions that are intrinsic to X as a complex manifold—namely the relation between the integral and de Rham cohomologies, and the decomposition of the de Rham cohomology in terms of holomorphic and anti-holomorphic forms.

2 Hodge theory for projective manifolds

The sorts of structures we see in the cohomology of an elliptic curve have analogues for an arbitrary complex projective manifold X, i.e.—a complex manifold that can be embedded as closed complex submanifold of some projective space:

$$\mathsf{X} \subset \mathbb{P}^N.$$

In general, the structure of the cohomology is not sufficient to determine X up to isomorphism, but nevertheless it is quite rich and carries a lot of information about X. We recall now the ingredients that define its "Hodge structure".

2.1 Cohomology groups

We begin by recalling several types of (co)homology that can be attached to a complex manifold X.

The singular homology $\mathsf{H}_\bullet(\mathsf{X};\mathbb{Z})$: it is defined by building a chain complex whose underlying $\mathbb{Z}$-module is freely generated by the set of maps from simplices to X. The singular cohomology $\mathsf{H}^\bullet(\mathsf{X};\mathbb{Z})$ is defined using the dual complex. We can also work with $\mathbb{C}$-coefficients instead of $\mathbb{Z}$-coefficients, in which case we have the vector space duality $\mathsf{H}^\bullet(\mathsf{X};\mathbb{C}) \cong \mathsf{H}_\bullet(\mathsf{X};\mathbb{C})^\vee$.

The de Rham cohomology $\mathsf{H}^\bullet_{\mathsf{dR}}(\mathsf{X})$: it is defined as the cohomology of the complex $(\mathcal{A}^\bullet(\mathsf{X}), \mathrm{d})$ of all C^∞ complex-valued differential forms on X, equipped with the de Rham differential $\mathrm{d} : \mathcal{A}^\bullet(\mathsf{X}) \to \mathcal{A}^{\bullet+1}(\mathsf{X})$.

The Dolbeault cohomology groups $\mathsf{H}^{p,q}(\mathsf{X})$: These are defined by decomposing the C^∞ forms into holomorphic and anti-holomorphic pieces, as follows.

We say that a form $\omega \in \mathcal{A}^\bullet(\mathsf{X})$ is of ***type*** (p, q) if in any local holomorphic coordinates $(z_1, \ldots, z_n)$, we may write ω as a sum of terms of the form

$$g \,\mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_p} \wedge \mathrm{d}\overline{z}_{j_1} \wedge \cdots \wedge \mathrm{d}\overline{z}_{j_q}$$

where g is an arbitrary smooth function. In other words every term of ω has p of the holomorphic forms $\mathrm{d}z$ and q of the anti-holomorphic forms $\mathrm{d}\overline{z}$. We denote by $\mathcal{A}^{p,q}(\mathsf{X}) \subset \mathcal{A}^{p+q}(\mathsf{X})$ the space of (p, q)-forms. We then have a direct sum decomposition

$$\mathcal{A}^n(\mathsf{X}) = \bigoplus_{p+q=n} \mathcal{A}^{p,q}(\mathsf{X})$$

and one can verify that the de Rham differential decomposes as a sum

$$\mathrm{d} = \partial + \overline{\partial}$$

where ∂ and $\overline{\partial}$ are the components of ∂ with respect to the two gradings:

$$\partial : \mathcal{A}^{\bullet,\bullet}(\mathsf{X}) \to \mathcal{A}^{\bullet+1,\bullet}(\mathsf{X})$$

and

$$\overline{\partial} : \mathcal{A}^{\bullet,\bullet}(\mathsf{X}) \to \mathcal{A}^{\bullet,\bullet+1}(\mathsf{X}).$$

The Dolbeault cohomology is then defined by taking the cohomology with respect to $\overline{\partial}$:

$$\mathsf{H}^{p,q}(\mathsf{X}) = \mathsf{H}^q(\mathcal{A}^{p,\bullet}(\mathsf{X}), \overline{\partial})$$

The sheaf cohomology groups $\mathsf{H}^q(\mathsf{X}, \Omega^p_{\mathsf{X}})$: A p-form ω is ***holomorphic*** if it can be written locally as a sum of terms of the form

$$g \,\mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_p}$$

where $(z_1, \ldots, z_n)$ are local holomorphic coordinates and g is a holomorphic function. These forms give a sheaf Ω^p_{X} on X, and we can take the sheaf cohomology groups $\mathsf{H}^q(\mathsf{X}, \Omega^p_{\mathsf{X}})$.

2.2 Relations between these cohomology groups

There are many relations between these cohomology groups when X is an arbitrary complex manifold. For instance:

Integral versus complex coefficients: The inclusion $\mathbb{Z} \hookrightarrow \mathbb{C}$ induces an isomorphism

$$\mathsf{H}^\bullet(\mathsf{X};\mathbb{C}) \cong \mathsf{H}^\bullet(\mathsf{X};\mathbb{C}) \otimes_{\mathbb{Z}} \mathbb{C}$$

Thus $\mathsf{H}^\bullet(\mathsf{X};\mathbb{Z})$ generates a $\mathbb{Z}$-submodule of the complex vector space $\mathsf{H}^\bullet(\mathsf{X};\mathbb{C})$.

The integration pairing: There is a natural pairing

$$\int : \mathsf{H}^\bullet_{\mathsf{dR}}(\mathsf{X}) \otimes \mathsf{H}_\bullet(\mathsf{X};\mathbb{C}) \to \mathbb{C}$$

defined by integration:

$$\omega \otimes \Delta \mapsto \int_\Delta \omega$$

where ω is a differential form and Δ is a simplex in X. The ***de Rham theorem*** states that this pairing is perfect, so that it induces an isomorphism

$$\mathsf{H}^\bullet_{\mathsf{dR}}(\mathsf{X}) \cong \mathsf{H}_\bullet(\mathsf{X};\mathbb{C})^\vee \cong \mathsf{H}^\bullet(\mathsf{X};\mathbb{C})$$

Sheaf versus Dolbeault cohomology: We introduced two types of cohomology groups that depended on a pair (p,q): the Dolbeault cohomology $\mathsf{H}^{p,q}(\mathsf{X})$ and the sheaf cohomology $\mathsf{H}^q(\mathsf{X},\Omega^p_{\mathsf{X}})$. In fact, these two spaces are canonically isomorphic:

$$\mathsf{H}^{p,q}(\mathsf{X}) \cong \mathsf{H}^q(\mathsf{X},\Omega^p_{\mathsf{X}}).$$

Let us sketch the proof of this ***Dolbeault isomorphism***.

The definition of (p,q)-forms is local, so instead of talking about the vector space of $\mathcal{A}^{p,q}(\mathsf{X})$ of global (p,q)-forms we can talk about the sheaf $\mathcal{A}^{p,q}_{\mathsf{X}}$. A form

$$\omega = g(z,\overline{z})\,\mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_p} \in \mathcal{A}^{p,0}_{\mathsf{X}}$$

is holomorphic if and only if g is a holomorphic function. This is equivalent to requiring that $\overline{\partial}\omega = 0$, which is a version of the Cauchy–Riemann equations.

Thus the sheaf Ω^p_{X} of holomorphic forms is the kernel

$$\Omega^p_{\mathsf{X}} = \ker \overline{\partial} : \mathcal{A}^{p,0}_{\mathsf{X}} \to \mathcal{A}^{p+1,0}_{\mathsf{X}}.$$

A version of the Poincaré lemma for the operator $\overline{\partial}$ says that the complex $(\mathcal{A}^{p,\bullet}_{\mathsf{X}}, \overline{\partial})$ of sheaves is exact in positive degrees, i.e. we have a quasi-isomorphism

$$\Omega^p_{\mathsf{X}} \cong (\mathcal{A}^{p,\bullet}_{\mathsf{X}}, \overline{\partial})$$

of complexes of sheaves. But $\mathcal{A}^{p,\bullet}_{\mathsf{X}}$ is the sheaf of sections of a C^∞ vector bundle, and hence it is an acyclic sheaf (i.e. it has no higher sheaf cohomology). We can therefore use the quasi-isomorphism above to compute the sheaf cohomology of $\Omega^\bullet_{\mathsf{X}}$ using the complex of global sections of $\mathcal{A}^{p,\bullet}_{\mathsf{X}}$, which is exactly the Dolbeault complex $(\mathcal{A}^{p,\bullet}(\mathsf{X}), \overline{\partial})$. This gives the Dolbeault isomorphism.

2.3 The Hodge decomposition

The definitions and results of the last two section hold for any complex manifold X. But notice that the decomposition we found for an elliptic curve, namely

$$\mathsf{H}^1_{\mathsf{dR}}(\mathsf{X}) \cong \mathsf{H}^{1,0}(\mathsf{X}) \oplus \mathsf{H}^{0,1}(\mathsf{X}),$$

cannot possibly hold for an arbitrary complex manifold X:

Example 2.1. If $\mathsf{X} = \mathbb{C}$ is the complex line, then $\mathsf{H}^1_{\mathsf{dR}}(\mathsf{X}) = 0$. On the other hand $\mathsf{H}^{1,0}(\mathsf{X})$ is the space of holomorphic one-forms, i.e. forms that can be written $f\,\mathrm{d}z$ where f is an entire holomorphic function on $\mathbb{C}$. Thus $\mathsf{H}^{1,0}(\mathsf{X})$ is infinite-dimensional. □

Note that the non-compactness is playing some role in this example, since the space of holomorphic functions (or forms) on a *compact* complex manifold is finite-dimensional.

However, there are also examples of compact complex manifolds for which the decomposition fails. Indeed, observe that in order for the inclusion $\mathcal{A}^{1,0}(\mathsf{X}) \to \mathcal{A}^1(\mathsf{X})$ to induce an inclusion $\mathsf{H}^{1,0}(\mathsf{X}) \hookrightarrow \mathsf{H}^1_{\mathsf{dR}}(\mathsf{X})$, it is necessary that every holomorphic one-form on X be closed. But this can fail:

Example 2.2. Consider the group of matrices

$$\mathsf{G} = \left\{ \begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix} \middle| z_1, z_2, z_3 \in \mathbb{C} \right\}$$

and let X be the quotient of G by the subgroup consisting of matrices for which $z_1, z_2, z_3 \in \mathbb{Z} + \mathrm{i}\mathbb{Z} \subset \mathbb{C}$. Then X is compact, and one can check that $\alpha = \mathrm{d}z_2 - z_3\mathrm{d}z_1$ descends to a holomorphic one-form on X, but clearly $\mathrm{d}\alpha = \mathrm{d}z_1 \wedge \mathrm{d}z_3 \neq 0$. This example was discovered by Kodaira; it is known as the ***Iwasawa manifold***. □

It is a deep theorem that these problems go away for projective manifolds:

Theorem 2.3 (Hodge decomposition)**.** *If X is a complex projective manifold (or more generally, a compact Kähler manifold), then the decomposition of C^∞-forms $\mathcal{A}^\bullet(\mathsf{X}) = \bigoplus_{p,q\geq 0} \mathcal{A}^{p,q}(\mathsf{X})$ induces a canonical decomposition*

$$\mathsf{H}^\bullet_{\mathsf{dR}}(\mathsf{X}) = \bigoplus_{p,q\geq 0} \mathsf{H}^{p,q}(\mathsf{X})$$

on cohomology. Moreover, complex conjugation gives a canonical isomorphism

$$\mathsf{H}^{q,p}(\mathsf{X}) \cong \overline{\mathsf{H}^{p,q}(\mathsf{X})}.$$

The proof is analytic in nature, using the techniques of elliptic PDE theory in an essential way. See, e.g. [3] for a detailed proof.

2.4 Pure Hodge structures and the Hodge filtration

The properties of the cohomology can be abstracted into the notion of a pure Hodge structure:

Definition 2.4. A ***pure Hodge structure of weight*** n is a $\mathbb{Z}$-module $\mathsf{H}_{\mathbb{Z}}$ whose complexification $\mathsf{H} := \mathsf{H}_{\mathbb{Z}} \otimes \mathbb{C}$ is equipped with a decomposition

$$\mathsf{H} = \bigoplus_{p+q=n} \mathsf{H}^{p,q}$$

such that $\overline{\mathsf{H}^{p,q}} = \mathsf{H}^{q,p}$.

Thus if X is a smooth projective variety, its nth cohomology defines a pure Hodge structure of weight n via the isomorphisms

$$\mathsf{H}^n(\mathsf{X};\mathbb{Z}) \otimes \mathbb{C} \cong \mathsf{H}^n_{\mathsf{dR}}(\mathsf{X}) \cong \bigoplus_{p+q=n} \mathsf{H}^{p,q}(\mathsf{X}).$$

Notice that the definition uses complex conjugation in an essential way, so in a sense the decomposition is not "holomorphic". More precisely, if we have a family X_b of projective manifolds depending homomorphically on a parameter $b \in \mathsf{B}$, then the cohomology $\mathsf{H}^n_{\mathsf{dR}}(\mathsf{X})$ forms a holomorphic vector bundle over the parameter space B, but the summands $\mathsf{H}^{p,q}(\mathsf{X}_b) \subset \mathsf{H}^n_{\mathsf{dR}}(\mathsf{X})$ do not give rise to holomorphic subbundles in general.

This can, in a sense, be remedied by working with filtrations instead of direct sums. Given a pure Hodge structure $\mathsf{H}_{\mathbb{Z}}$ of weight n, we can define the ***Hodge filtration***

$$\mathsf{H} \supset \cdots \supset \mathsf{F}^{p-1}\mathsf{H} \supset \mathsf{F}^p\mathsf{H} \supset \mathsf{F}^{p+1}\mathsf{H} \supset \cdots$$

where

$$\mathsf{F}^p\mathsf{H} = \bigoplus_{p'+q=n, p' \geq p} \mathsf{H}^{p',q}$$

Evidently we can recover the decomposition using complex conjugation, by taking the intersection:

$$\mathsf{H}^{p,q} = \mathsf{F}^p\mathsf{H} \cap \overline{\mathsf{F}^q\mathsf{H}}$$

Hence we can define pure Hodge structures using filtrations instead:

Definition 2.5 (Equivalent to Definition 2.4)**.** A ***pure Hodge structure of weight*** n is a $\mathbb{Z}$-module $\mathsf{H}_{\mathbb{Z}}$ whose complexification $\mathsf{H} := \mathsf{H}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ is equipped with a decreasing filtration $\mathsf{F}^\bullet\mathsf{H}$ such that

$$\mathsf{F}^p\mathsf{H} \cap \overline{\mathsf{F}^q\mathsf{H}} = 0 \qquad \text{and} \qquad \mathsf{F}^p\mathsf{H} \oplus \overline{\mathsf{F}^q\mathsf{H}} = \mathsf{H}$$

whenever $p + q = n + 1$.

Now if we have a family X_b depending holomorphically on $b \in \mathsf{B}$, the subspaces $\mathsf{F}^p\mathsf{H}^n_{\mathsf{dR}}(\mathsf{X}_b) \subset \mathsf{H}^n_{\mathsf{dR}}(\mathsf{X}_b)$ form holomorphic subbundles over B. Moreover, the notion of a Hodge *filtration* can be extended to the singular, noncompact and noncommutative settings.

2.5 Algebraic construction of the Hodge decomposition

Using filtrations one can give a completely algebraic construction of the Hodge structure. Let us now consider our complex projective manifold X as an algebraic subvariety of $\mathbb{P}^N$, equipped with the Zariski topology, and let $\Omega^\bullet_{\mathsf{X}}$ be the sheaf of algebraic (rather than holomorphic) differential forms. It carries the usual de Rham differential. A deep result of Grothendieck shows that we can calculate the de Rham cohomology using this purely algebro-geometric complex:

Theorem 2.6 (Grothendieck's algebraic de Rham theorem [4])**.** *Let X be a smooth algebraic variety over $\mathbb{C}$, and let X_{an} be the corresponding complex manifold. Then the hypercohomology of the complex $(\Omega^\bullet_{\mathsf{X}}, \mathrm{d})$ of sheaves in the Zariski topology computes the usual de Rham cohomology of X_{an}:*

$$\mathsf{H}^\bullet(\mathsf{X}, (\Omega^\bullet_{\mathsf{X}}, \mathrm{d})) \cong \mathsf{H}^\bullet_{\mathsf{dR}}(\mathsf{X}_{\mathsf{an}})$$

Notice that we have a filtration on the algebraic de Rham complex:

$$\mathsf{F}^p\Omega^\bullet_{\mathsf{X}} = \left(0 \longrightarrow \Omega^p_{\mathsf{X}} \longrightarrow \Omega^{p+1}_{\mathsf{X}} \longrightarrow \cdots \right) \subset \Omega^\bullet_{\mathsf{X}}$$

This filtration induces a spectral sequence for the hypercohomology, with E_1-page

$$\mathsf{E}_1^{p,q} = \mathsf{H}^q(\mathsf{X}, \Omega^p_{\mathsf{X}}) \implies \mathsf{H}^{p+q}_{\mathsf{dR}}(\mathsf{X})$$

This spectral sequence is known as the ***Hodge–de Rham spectral sequence***.

The following result can be deduced from the original Hodge decomposition for complex manifolds, but a purely algebraic proof is also possible using positive characteristic techniques, as established by Faltings [2] and Deligne–Illusie [1]:

Theorem 2.7. *If X is a smooth and proper algebraic variety over a field of characteristic zero, then its Hodge–de Rham spectral sequence degenerates at the E_1 page.*

Recall that the degeneration of the spectral sequence at E_1 means that for each n we have a natural filtration $\mathsf{F}^\bullet\mathsf{H}^n_{\mathsf{dR}}(\mathsf{X})$ whose associated graded is the direct sum $\bigoplus_{p+q=n} \mathsf{H}^q(\mathsf{X}, \Omega^p_{\mathsf{X}})$. Thus there exists an isomorphism

$$\mathsf{H}^\bullet_{\mathsf{dR}}(\mathsf{X}) \cong \bigoplus_{p,q} \mathsf{H}^q(\mathsf{X}, \Omega^p_{\mathsf{X}})$$

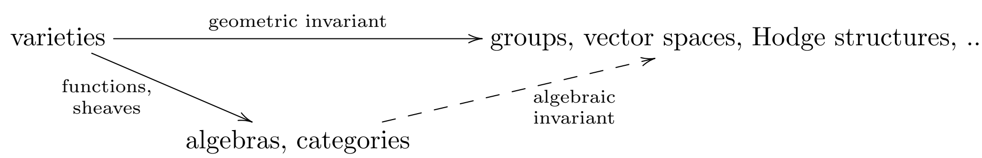
but from this the statement of Theorem 2.7 it is not obvious that the decomposition is canonical.

3 Noncommutative Hodge structures

Associated to an algebraic variety X are various algebraic structures, such as the ring $\mathcal{O}(\mathsf{X})$ of functions, or the category $\mathsf{QCoh}(\mathsf{X})$ of quasi-coherent sheaves. The point of algebraic geometry is that we can study the geometry of X using the algebraic properties of $\mathcal{O}(\mathsf{X})$ and $\mathsf{QCoh}(\mathsf{X})$. In fact, much more is true. When X is affine, we can recover X from its commutative algebra $\mathcal{O}(\mathsf{X})$ of functions (basically by definition). This fails when X is not affine, as the algebra of global functions is too small, but a deeper result, known as the Gabriel–Rosenberg reconstruction theorem, says that X can always be completely recovered from the abelian category $\mathsf{QCoh}(\mathsf{X})$.

The guiding principle of noncommutative algebraic geometry is that we should imagine that an arbitrary associative algebra (or category) is the ring of functions (or category of sheaves) on some fictitious "noncommutative algebraic variety". We can then try to use our intuition and techniques from geometry to get a handle on more abstract algebras and categories.

For instance, we may want to extend the various cohomology theories for varieties that we have been discussing to give invariants of abstract algebras and categories. In other words we seek a commutative diagram



Indeed, it turns out that there is a fairly precise table of analogies:

Invariant of a variety X	Invariants of a category $\mathcal{C}$
Dolbeault cohomology $\mathsf{H}^{\bullet,\bullet}(\mathsf{X})$	Hochschild homology $\mathsf{HH}_{\bullet}(\mathcal{C})$
de Rham cohomology $\mathsf{H}^{\bullet}_{\mathsf{dR}}(\mathsf{X})$	Periodic cyclic homology $\mathsf{HP}_{\bullet}(\mathcal{C})$
Hodge filtration $\mathsf{F}^{\bullet}\mathsf{H}^{\bullet}_{\mathsf{dR}}(\mathsf{X})$	Noncommutative Hodge filtration $\mathsf{F}^{\bullet}\mathsf{HP}_{\bullet}(\mathcal{C})$
Integral cohomology $\mathsf{H}^{\bullet}(\mathsf{X},\mathbb{Z})$	Topological K-theory $\mathsf{K}^{\bullet}(\mathcal{C})$
Lattice $\mathsf{H}^{\bullet}(\mathsf{X};\mathbb{Z}) \to \mathsf{H}^{\bullet}_{\mathsf{dR}}(\mathsf{X})$	Chern character $\mathsf{K}^{\bullet}(\mathcal{C}) \to \mathsf{HP}_{\bullet}(\mathcal{C})$
$\vdots$	$\vdots$

The invariants on the right typically have a bit less structure than those on the left. For instance, while de Rham cohomology is $\mathbb{Z}$-graded by the degree of differential forms, periodic cyclic homology only has a $\mathbb{Z}/2\mathbb{Z}$-grading, corresponding to the forms of even and odd degree.

The objects on the right are expected to assemble into a "noncommutative Hodge structure", a notion introduced by Katzarkov–Kontsevich–Pantev [6]. We will spend much of our time in this seminar simply introducing the ingredients on the right hand side of this table, and exploring their basic properties. This will lead naturally to one of the deep results of noncommutative Hodge theory: the noncommutative analogue of the Hodge–de Rham degeneration theorem (Theorem 2.7). This was conjectured by Kontsevich–Soibelman [7] and proven by Kaledin [5]. The proof is inspired by Deligne–Illusie's proof in the commutative case, and as such it uses reduction to positive characteristic.

In closing, let us mention a couple of potential applications which have motivated the study of noncommutative Hodge structures:

Mirror symmetry: String theory predicts the existence of “mirror pairs” of Kähler manifolds X and X', such that the symplectic geometry of X is “equivalent” to the algebraic geometry of X'. For instance, the Gromov–Witten invariants that count holomorphic curves in X should be computable from Hodge-theoretic data constructed from X'. Kontsevich proposed that mirror symmetry can be phrased as an equivalence between two categories: the derived Fukaya category of the symplectic manifold X and the derived category of coherent sheaves of the complex manifold X'. Then various geometric/numerical consequences predicted by physics can be explained by comparing the Hodge structures of these two isomorphic categories.

Deformation quantization: Deformation quantization is the study of the noncommutative deformations of a classical smooth variety X. In other words, we deform the product on $\mathcal{O}_{\mathsf{X}}$ to some new product $\star$:

$$f \star g = fg + \hbar B_1(f,g) + \hbar^2 B_2(f,g) + \cdots$$

where $\hbar$ is a deformation parameter.

At first order, such a deformation is determined by a geometric object known as a Poisson bracket, and Kontsevich [8] proved that there is a canonical way to extend such a first-order deformation to all orders. In fact he gave an explicit formula for a “quantization map”

$$\{\text{Poisson manifolds}\} \to \{\text{noncommutative algebras}\}$$

and proved that it gives an equivalence between Poisson brackets and noncommutative deformations at the level of formal derived deformation theory.

In principle, this allows one to study many interesting noncommutative algebras using the purely geometric techniques of Poisson geometry. However, the formula is highly transcendental; it is given by a complicated formal power series whose convergence is unknown, and it seems to be extremely difficult to compute the quantization explicitly in all but the very simplest examples. So in practice, the full power of the formula has yet to be realized. Nevertheless, Kontsevich has argued that in many interesting examples, the relevant noncommutative Hodge structure *is* computable, and that this allows one to explicitly determine the deformed algebra without directly summing the series.

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