Quantum Groups and Link Invariants

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1 Introduction

These notes are part of a seminar on topological field theories at the University of Edinburgh. In particular, this lecture gives a first example of quantum groups and shows one of their applications; knot invariants. It should be noted that this lecture will not make use of the definition of a topological field theory and it may look out of place in the seminar but we will see how it is connected in the following lecture. Unless otherwise stated, the reference for all this material is Kassel's *Quantum Groups* [1].

2 Quantum Groups

Unfortunately, there is no one definition for quantum groups and the term instead refers to various classes of objects, usually noncommutative algebras with some sort of additional structure. One such class consists of deformations of universal enveloping algebras of lie algebras and the specific example we will consider is a deformation of the universal enveloping algebra of $\mathfrak{sl}_2$. This is an example of a Hopf Algebra and so we begin by defining those.

2.1 Hopf Algebras

A Hopf Algebra is, in particular, an algebra and so we start with the definition of an algebra.

Definition 2.1. *An associative algebra over a field k is given by a triple (A, μ, η) where A is a k-vector space and the k-linear maps $\mu : A \otimes A \to A$ and $\eta : k \to A$ are such that the following diagrams commute.*

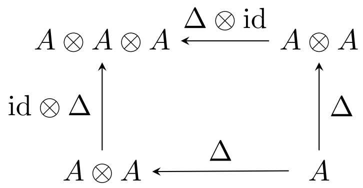
$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mu \otimes \mathrm{id}} & A \otimes A \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mathrm{id} \otimes \mu} \\ A \otimes A & \xrightarrow{\mu} & A \end{array} \qquad \begin{array}{ccccc} k \otimes A & \xrightarrow{\eta \otimes \mathrm{id}} & A \otimes A & \xrightarrow{\mathrm{id} \otimes \eta} & A \otimes k \\ & {\scriptstyle \cong}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle \cong} & \\ & & A & & \end{array}$$

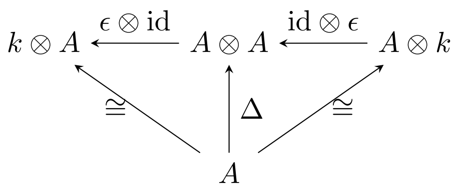
The first diagram gives the associativity of the algebra and second diagram shows the algebra is unital. Moreover, if we wish the algebra A to be commutative we ask that the following diagram also commutes, where $\tau_{A,A}(a \otimes b) = b \otimes a$.

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\tau_{A,A}} & A \otimes A \\ & {\scriptstyle \mu}\searrow \quad \swarrow{\scriptstyle \mu} & \\ & A & \end{array}$$

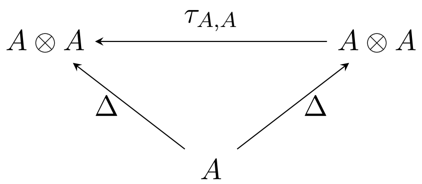
From this definition of algebra, it is very easy to define a coalgebra simply by reversing all the arrows. However, to be explicit we give the following definition.

Definition 2.2. *A coassociative coalgebra over a field k is given by a triple (A, Δ, ϵ) where A is a k-vector space and the k-linear maps $\Delta : A \to A \otimes A$ and $\epsilon : A \to k$ are such that the following diagrams commute.*





The two diagrams give the coassociativity of the coalgebra and the fact the coalgebra is counital. Moreover, if we wish the coalgebra A to be cocommutative we ask that the following diagram also commutes.



We call the maps Δ and ϵ the coproduct and counit of the coalgebra respectively. Given any two algebras, or coalgebras, we can define the notion of a morphism between them.

Definition 2.3.

1. *An algebra morphism between two algebras (A, μ_A, η_A) and (B, μ_B, η_B) is a linear map $f : A \to B$ such that $\mu_B \circ (f \otimes f) = f \circ \mu_A$ and $f \circ \eta_A = \eta_B$.*
2. *A coalgebra morphism between two coalgebras $(A, \Delta_A, \epsilon_A)$ and $(B, \Delta_B, \epsilon_B)$ is a linear map $f : A \to B$ such that $(f \otimes f) \circ \Delta_A = \Delta_B \circ f$ and $\epsilon_B \circ f = \epsilon_A$.*

This allows us to define a bialgebra.

Definition 2.4. *A bialgebra is a quintuple $(A, \mu, \eta, \Delta, \epsilon)$ such that (A, μ, η) is an algebra and (A, Δ, ϵ) is a coalgebra with the additional requirement that Δ and ϵ are both morphisms of algebras.*

Note that we could have equivalently asked that μ and η were coalgebra morphisms. A common example of a bialgebra is the group algebra of a finite group where $\Delta(g) = g \otimes g$ and $\epsilon(g) = 1$ for all $g \in G$. A Hopf Algebra is a bialgebra with some additional structure.

Definition 2.5.

1. *Let $(A, \mu, \eta, \Delta, \epsilon)$ be a bialgebra. A linear map $S : A \to A$ is called an antipode for A if*

$$\mu \circ (S \otimes \mathrm{id}_A) \circ \Delta = \mu \circ (\mathrm{id}_A \otimes S) \circ \Delta = \eta \circ \epsilon$$

2. *A Hopf algebra is a bialgebra with an antipode.*

Note that not all bialgebras have an antipode so Hopf Algebras are a special class of bialgebras. Moreover, the antipode will be an antihomorphism i.e. $S(ab) = S(b)S(a)$ for all $a, b \in A$.

Now we turn our attention to the deformation of the universal enveloping algebra of $\mathfrak{sl}_2$, which will be the focus of this lecture. Recall that $\mathfrak{sl}_2$ is the 3-dimensional lie algebra generated by

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with relations

$$[E, F] = H, \quad [H, E] = 2E \quad \text{and} \quad [H, F] = -2F.$$

We are interested in the representation theory of $\mathfrak{sl}_2$ i.e. the $\mathfrak{sl}_2$-modules. However, lie algebras are not associative and so we can not use the array of tools developed for studying the representation theory of associative unital algebras. Therefore, when studying a lie algebra $\mathfrak{g}$, we often choose to study an associative unital algebra which in some sense has “the same” representation theory as our original lie algebra. Such an algebra can be chosen in a universal way and is called the universal enveloping algebra of $\mathfrak{g}$, denoted $U(\mathfrak{g})$.

Note that $U(\mathfrak{sl}_2)$ is a cocommuative bialgebra with coproduct

$$\Delta(x) = 1 \otimes x + x \otimes 1 \quad \forall x \in \mathfrak{sl}_2.$$

However, for reasons which will become clear later, we choose to study the deformed algebra $U_q(\mathfrak{sl}_2)$ which can be defined as follows. Pick $q \in \mathbb{C}^\times$ such that q is not a root of unity. Then $U_q(\mathfrak{sl}_2)$ is the $\mathbb{C}(q)$-algebra generated by E, F, K, K^{-1}, with relations

$$KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F \quad \text{and} \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

To motivate these relations, note that the generators act on the irreducible two dimensional representation of $U_q(\mathfrak{sl}_2)$ as

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad K = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$$

and that these matrices satisfy the relations above. In fact, we can think of K as q^H and if we take a certain limit as $q \to 1$ we get the original $U(\mathfrak{sl}_2)$ back.

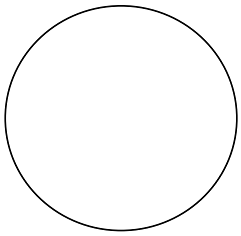
The Hopf Algebra structure on $U_q(\mathfrak{sl}_2)$ is given by

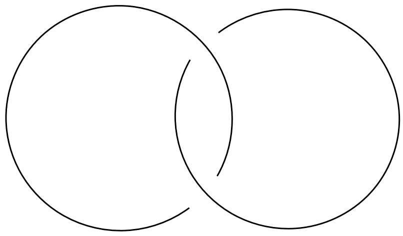
$$\begin{aligned} \Delta(E) = 1 \otimes E + E \otimes K; \quad & \Delta(F) = K^{-1} \otimes F + F \otimes 1; \\ \Delta(K) = K \otimes K; \quad & \Delta(K^{-1}) = K^{-1} \otimes K^{-1}; \\ \epsilon(E) = \epsilon(F) = 0, \quad & \epsilon(K) = \epsilon(K^{-1}) = 1 \\ S(E) = -EK^{-1}, \ S(F) = -KF, \quad & S(K) = K^{-1}, \ S(K^{-1}) = K. \end{aligned}$$

In particular, note that $U_q(\mathfrak{sl}_2)$ is not cocommutative which, although it seems like we are making things more complicated, is precisely what allows us to obtain knot invariants.

3 Knot Theory Basics

Knot theory is essentially studying what we get when we take a piece of string, tangle it up and then tie the ends together. We are actually going to look at a generalisation of knots, called links, in which more than one piece of string is allowed. A more mathematical definition is as follows.


(a) Unknot


(b) Two linked circles

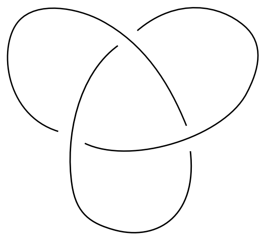

(c) Trefoil

Figure 1: Examples of links.

Definition 3.1. *A link is a collection of finitely many circles smoothly embedded in* $\mathbb{R}^3$*. A knot is a link consisting of a single circle.*

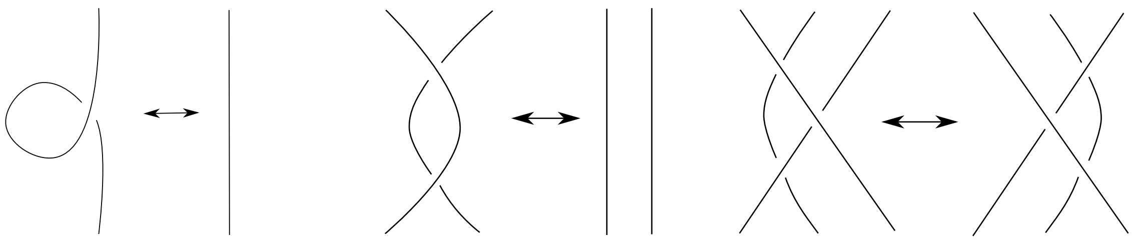
Examples are shown in Figure 1. Note that (a) and (c) are examples of knots where as (b) is only a link.

The fundamental question of knot theory is to ask when two knots are "the same". Intuitively, two links are the same if you can get from one to the other without cutting the string anywhere.

Definition 3.2. *Two links are considered isotopic if there exists an isotopy of* $\mathbb{R}^3$ *which maps one link to the other.*

Figure 1 shows that we can draw links in the plane by keeping track of whether crossings are over-crossings or under-crossings. However, there are multiple ways of drawing the same link so we would like to know when two such diagrams represent the same link.

Proposition 3.3. *Two links are isotopic if and only if you can change from one to the other using the following* Reidemeister moves*:*



Despite this very useful characterisation, it can still be very difficult to tell whether or not two diagrams represent the same link. As a tool to help prove that two links are different, mathematicians have developed various link invariants.

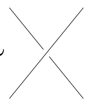
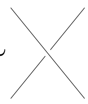
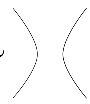
Definition 3.4. *A link invariant assigns to each link an object such that, if two links are isotopic, they are assigned the same object.*

These are useful because, if two different objects are assigned to two links, we know they can not be isotopic. A simple example is to assign to each link the number of circles which make up the link. It's clear that this is a link invariant but unfortunately it's not a very useful one. For example, it can't tell the difference between the unknot and the trefoil in Figure 1. Therefore, mathematicians looked for more sophisticated invariants and, since the 1920's, they have been assigning polynomials as link invariants. One such example is the Jones Polynomial.

Example 3.5. Let L be a link. We define the Jones Polynomial, $P_L(t)$, of L inductively.

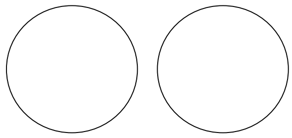
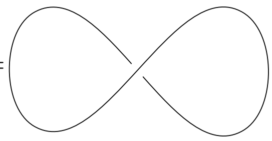
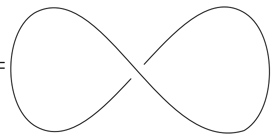
- If L is the unknot then $P_L(t) = 1$.

- If L_+, L_- and L_0 are three links, identical except at a single crossing point where

  $L_+ \sim$  $L_- \sim$  $L_0 \sim$ 

  then $t^{-1}P_{L_+}(t) - tP_{L_-}(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})P_{L_0}(t)$. This is called the Skein relation of the Jones Polynomial.

As an example, we find the Jones Polynomial of the link consisting of two unlinked circles. We define L_0 to be our desired link and the others as follows:

$L_0 =$  $L_+ =$  $L_- =$ 

We can see that both L_+ and L_- are isotopic to the unknot and so we get

$$(t^{\frac{1}{2}} - t^{-\frac{1}{2}})P_{L_0}(t) = t^{-1} - t \quad \text{and so} \quad P_{L_0}(t) = \frac{t^{-1} - t}{(t^{\frac{1}{2}} - t^{-\frac{1}{2}})}.$$

Since we can always put a link in terms of simpler links using this inductive method, we can calculate the Jones Polynomial of any link. To show it is a link invariant, you just need to show that it doesn't change when we alter a link by any of the Reidemeister moves.

The goal of the rest of the lecture is to construct link invariants, including the Jones Polynomial, using the category $U_q(\mathfrak{sl}_2)$-mod. The idea is that we will relate morphisms in $U_q(\mathfrak{sl}_2)$-mod to tangles. Tangles are a generalisation of links where we don't require the two ends of the string to be tied together.

Definition 3.6. *A tangle is a smooth embedding of arcs and circles into $\mathbb{R}^2 \times I$ where the endpoints of the arcs lie on $\mathbb{R}^2 \times \partial I$.*

Note that a link can be viewed as a tangle with only circles embedded. As with links, tangles can be drawn in the plane and two diagrams represent the same tangle if and only if they are related by the Reidemeister moves.

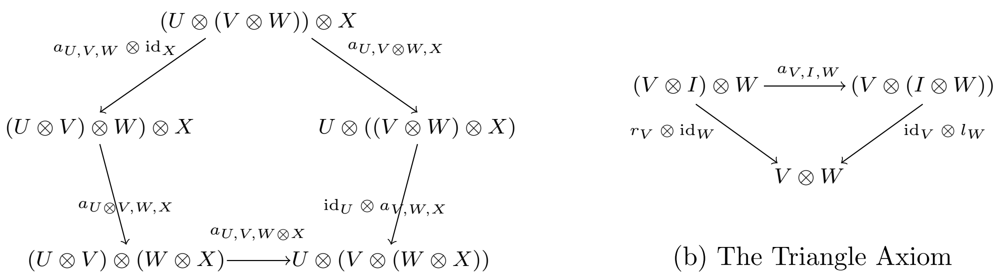
We are going to relate to each tangle a morphism in $U_q(\mathfrak{sl}_2)$-mod and, in particular, to each link, a morphism $\mathbb{C}(q) \to \mathbb{C}(q)$. This map can be thought of as an element of $\mathbb{C}(q)$ and will be the link invariant. The reason this is going to work is because $U_q(\mathfrak{sl}_2)$ is a Hopf Algebra which ensures $U_q(\mathfrak{sl}_2)$-mod is a balanced, rigid, braided tensor category which we discuss in more detail now.

4 Tensor Categories

Definition 4.1. *A tensor category is a category $\mathcal{C}$ with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and:*

- *A natural isomorphism $a : \otimes(\otimes \times \mathrm{id}) \to \otimes(\mathrm{id} \times \otimes)$ called the associativity constraint;*
- *An object $I \in \mathcal{C}$ called the unit and natural isomorphisms $l : \otimes(I \times \mathrm{id}) \to \mathrm{id}$ and $r : \otimes(\mathrm{id} \times I) \to \mathrm{id}$ called the left and right unit constraints respectively,*

such that the Pentagon and Triangle Axioms hold i.e. the two diagrams in Figure 3 commute for all objects $U, V, W, X \in \mathcal{C}$.



(a) The Pentagon Axiom

(b) The Triangle Axiom

Figure 3: The Pentagon and Triangle Axiom for Definition 4.1.

Example 4.2. 1. The obvious example of a tensor category is the category of k-vector spaces, denoted $\mathrm{Vect}(k)$, with the usual tensor product. Here, the unit is given by k.

2. Over a k-algebra A, every A-module is a k-vector space and so A-mod is a subcategory of $\mathrm{Vect}(k)$. Thus, it will inherit the tensor category structure from $\mathrm{Vect}(k)$ if, for any two A-modules V and W, we can give the vector space $V \otimes W$ an action of A. For a general k-algebra there is no canonical way to do this but if A has a coproduct $\Delta : A \to A \otimes A$ (such as $U_q(\mathfrak{sl}_2)$), we can define the action as

$$a \cdot (v \otimes w) = \Delta(a)(v \otimes w).$$

Thus $U_q(\mathfrak{sl}_2)$-mod is a tensor category.

As discussed, our link invariants are going to come from relating morphisms in $U_q(\mathfrak{sl}_2)$ to tangles. In Figure 4, we begin introducing how we might draw these morphisms to make this connection. We emphasise that the tensor product of two morphisms is simply the two morphisms drawn next to each other and composition by a morphism g means adding the picture related to g on top of the original morphism.

The definition of tensor category had a notion of associativity built into it but we would also like a notion of commutativity as this often makes structures easier to work with.

Definition 4.3. *1. Define the flip functor, $\tau : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$, such that $\tau(V, W) = (V, W)$.*

2. A braided tensor category is a tensor category $(\mathcal{C}, \otimes)$ with a natural isomorphism

$$c : \otimes \to \otimes \circ \tau$$

called the commutativity constraint satisfying the Hexagon Axiom.

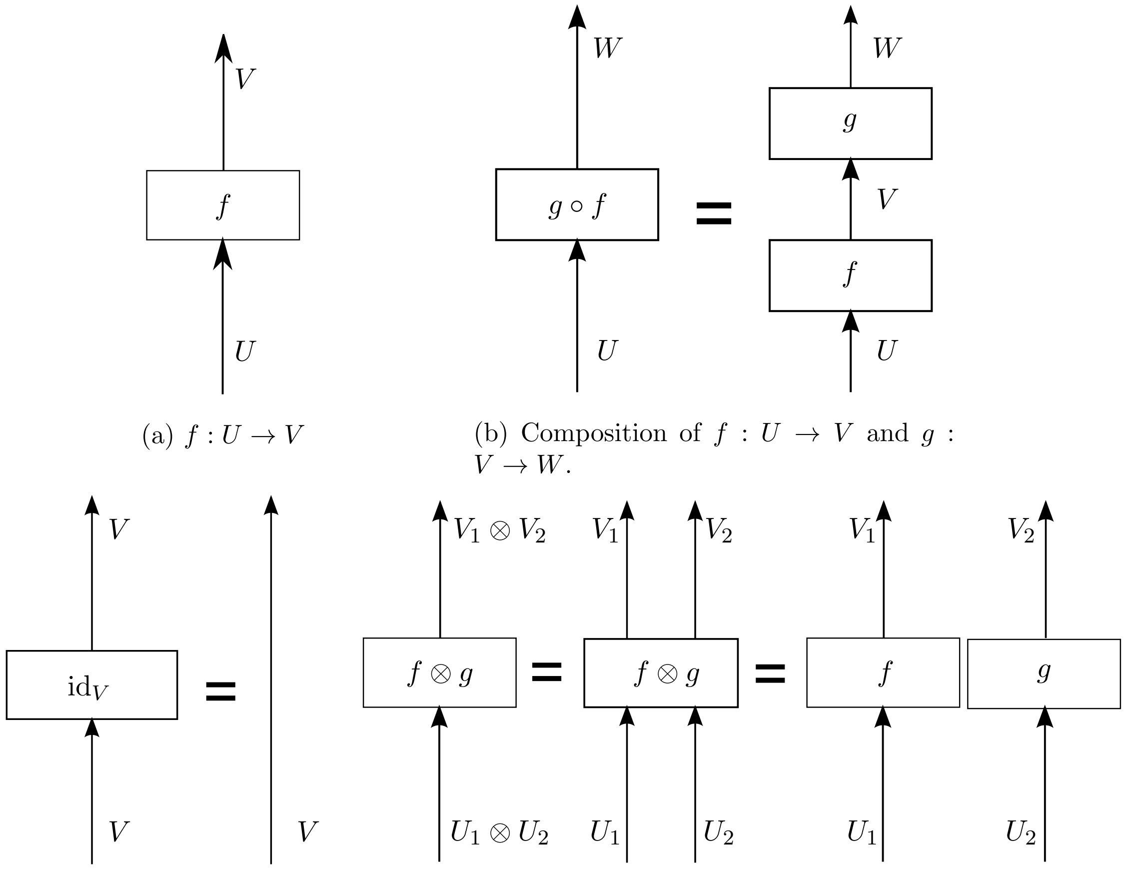
Note that this just means we have an isomorphism $c_{V,W} : V \otimes W \to W \otimes V$ for every $V, W \in \mathcal{C}$ satisfying some compatibility relations.

Example 4.4. 1. In $\mathrm{Vect}(k)$, we can take c such that $c_{V,W}(v \otimes w) = w \otimes v$. Notice that $c_{W,V} \circ c_{V,W} = \mathrm{id}_{V \otimes W}$ for all $V, W \in \mathcal{C}$ and so we say $\mathrm{Vect}(k)$ is a symmetric braided tensor category.

2. In $U_q(\mathfrak{sl}_2)$-mod, the commutativity constraint from $\mathrm{Vect}(k)$ is not compatible with the action of $U_q(\mathfrak{sl}_2)$ and so we need to look for a different braiding.

Definition 4.5. *In a bialgebra A, a universal R-matrix is an invertible element of $A \otimes A$ such that*

$$\tau_{A,A} \circ \Delta(x) = R\Delta(x)R^{-1} \quad \forall x \in A.$$



(a) $f : U \to V$

(b) Composition of $f : U \to V$ and $g : V \to W$.

(c) $\mathrm{id}_V : V \to V$

(d) Tensor product of $f : U_1 \to V_1$ and $g : U_2 \to V_2$.

Figure 4: Morphisms in a tensor category.

If an R-matrix exists we can define a braiding in A-mod by

$$c_{V,W}(v \otimes w) = \tau_{V,W}(R(v \otimes w)).$$

where $\tau_{V,W}(v \otimes w) = w \otimes v$. If A is a cocommutative bialgebra, then you can take the R-matrix to be $1 \otimes 1$ and A-mod becomes a symmetric braided tensor category with c as in $\mathrm{Vect}(k)$. However, when A is not cocommutative, these R-matrices may not exist and can be very difficult to find even when they do. For $U_q(\mathfrak{sl}_2)$, you need to pass to the completion to find the R-matrix which has the expression

$$R = q^{\frac{H \otimes H}{2}} \exp_q(q - q^{-1}) E \otimes F).$$

More information about this can be found [2]. Even though this R-matrix only exists in the completion, it is enough to give us a well defined braiding on $U_q(\mathfrak{sl}_2)$-mod. As an example, if we consider the 2 dimensional irreducible representation of $U_q(\mathfrak{sl}_2)$, $V = \mathbb{C}^2$ with standard basis $\{e_1, e_2\}$, then we get

$$c_{V,V} = q^2 \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

with respect to the basis $\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\}$. Note that $c_{V,V}(e_1 \otimes e_1) = q^3 e_1 \otimes e_1$ and so it is clear that $c^2_{V,V} \neq \mathrm{id}_{V \otimes V}$ and hence $U_q(\mathfrak{sl}_2)$-mod is not a symmetric tensor category.

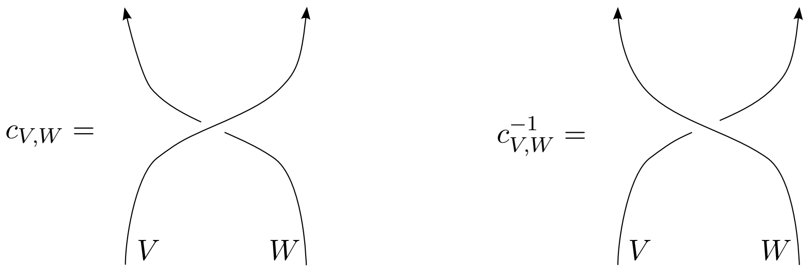


Figure 5: Representing the braiding isomorphisms as tangles.

While not being symmetric might make the category more complicated, it is exactly what we need to get useful link invariants. We represent the braiding isomorphisms in our category by the tangles shown in Figure 5 and, in particular, notice that if our category is symmetric, we see that we don't care whether or not a strand crosses above or below another. For example, this means we would be unable to tell the difference between two linked circles and two unlinked circles. In particular, a knot invariant from a symmetric category would assign the same object to every knot and so be useless as a knot invariant. Thus, it is the non-cocommutativity of $U_q(\mathfrak{sl}_2)$ and so the lack of symmetry in $U_q(\mathfrak{sl}_2)$-mod which will give interesting knot invariants.

Recall that every vector space has a dual and this gives extra structure to the category $\mathrm{Vect}(k)$.

Definition 4.6. *A braided tensor category is rigid if every object V has a dual object, denoted V^*, and morphisms*

$$e_V : V^* \otimes V \to I$$
$$i_V : I \to V \otimes V^*$$

such that

$$(\mathrm{id}_V \otimes e_V)(i_V \otimes \mathrm{id}_V) = \mathrm{id}_V \quad \textit{and} \quad (e_V \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^*} \otimes i_V) = \mathrm{id}_{V^*}.$$

Example 4.7.

1. In $\mathrm{Vect}(k)$, we have $V^* = \mathrm{Hom}_k(V, k)$ and if V has basis $\{v_1, \ldots, v_n\}$, and the dual basis is $\{v^1, \ldots, v^n\}$ then we have

$$e_V(v^i \otimes v_j) = v^i(v_j)$$
$$i_V(1) = \sum_i v_i \otimes v^i$$

   which are called evaluation and coevaluation respectively.

2. As with the tensor category structure, the duals in $U_q(\mathfrak{sl}_2)$-mod and the corresponding maps are inherited from $\mathrm{Vect}(\mathbb{C})$, provided that we can define an action of $U_q(\mathfrak{sl}_2)$ on V^* such that e_V and i_V are both module homomorphisms. For this to hold, we need the antipode $S : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2)$ which exists as $U_q(\mathfrak{sl}_2)$ is a Hopf Algebra. The action of $U_q(\mathfrak{sl}_2)$ on V^* is then given by

$$(a \cdot f)(v) = f(S(a) \cdot v) \quad \forall f \in V^*.$$

Now that we have these duals and these extra morphisms, we wish to know how to represent them as tangles. As we don't want to clutter up our tangles with lots of notation, if an arrow should be labelled by V^*, we instead label it by V but give the arrow the opposite orientation.

This is shown in Figure 6. Also shown in the diagram is how we draw e_V and i_V. We consider the unit as "the empty object" of the category and so if an arrow should be labelled by I, we just don't draw it. This is the key point, as drawing them like this is going to let us view links as morphisms from $\mathbb{C}(q)$ to $\mathbb{C}(q)$.

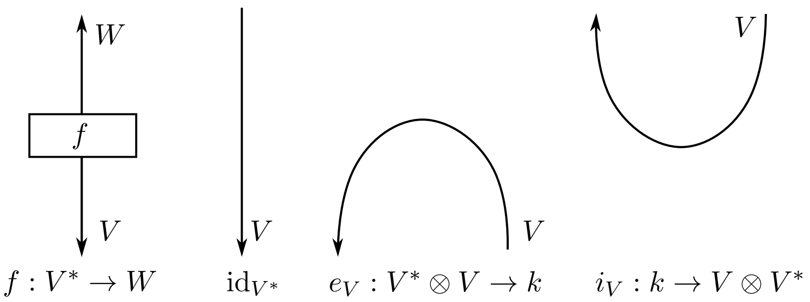
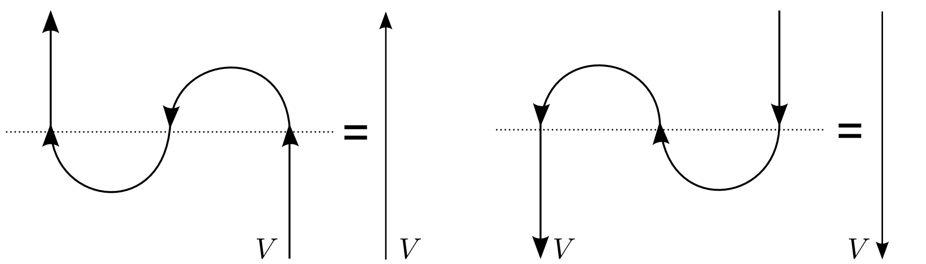


Figure 6: Pictorial representations of the morphisms containing duals.

Before we go any further, we give an example of using the graphical representation of morphisms. For example, we can use them to express the conditions given in Definition 4.6 as shown in Figure 7. It is clear by the Reidemeister moves that the two tangles in each diagram are the same and so should represent the same morphism.



(a) $(\mathrm{id}_V \otimes e_V)(i_V \otimes \mathrm{id}_V) = \mathrm{id}_V$ (b) $(e_V \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^*} \otimes i_V) = \mathrm{id}_{V^*}$

Figure 7: Pictorial representations of the conditions in Definition 4.6.

To get a link invariant we are going to construct a functor from a category where tangles are the morphisms to the category $U_q(\mathfrak{sl}_2)$-mod. For this to be well defined we need two things:

1. A way of associating to any tangle a morphism in $U_q(\mathfrak{sl}_2)$-mod.

2. If two tangles are isotopic we need that the two morphisms they represent are the same.

Unfortunately neither of these are satisfied at the moment. First, note that strands of tangles will have to be labelled by an object of $U_q(\mathfrak{sl}_2)$-mod for us to have any hope of knowing which morphism to send the tangle to. This problem will be solved by requiring that the category whose morphisms are tangles is in fact a category of "coloured" tangles. This precisely means that each strand of each tangle is coloured with an object of $U_q(\mathfrak{sl}_2)$-mod. However, even with this addition, we can't satisfy either of the above conditions as the following example shows.

Example 4.8. At this point, we have no way of associating a morphism to the top cap of the tangle in Figure 8. This would need to be a morphism $V \otimes V^* \to I$ which we don't even know exists at the moment. However, again we are saved by the extra structure that $U_q(\mathfrak{sl}_2)$-mod has.

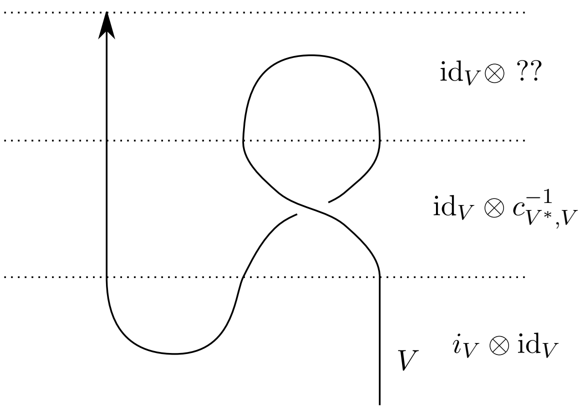


Figure 8: While we can associate morphisms to most of this tangle, we don't currently know what to assign to the top cap. Moreover, by Reidemister moves, this tangle is isotopic to the straight line corresponding to id_V and so the morphism it represents should be id_V.

Definition 4.9. 1. *The transpose of $f : U \to V$ is $f^* : V^* \to U^*$ where*

$$f^* = (e_v \otimes \mathrm{id}_{U^*})(\mathrm{id}_{V^*} \otimes f \otimes \mathrm{id}_{U^*})(\mathrm{id}_{V^*} \otimes i_U).$$

2. *A braided tensor category $(\mathcal{C}, \otimes)$ is balanced if there exists a natural isomorphism*

$$\theta : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$$

such that

$$\theta_{V\otimes W} = (\theta_V \otimes \theta_W)c_{W,V}c_{V,W} \quad \textit{and} \quad \theta_{V^*} = (\theta_V)^*.$$

3. *A ribbon category is a rigid, balanced, braided tensor category.*

Example 4.10. 1. Any symmetric braided tensor category is balanced with $\theta_V = \mathrm{id}_V$. In particular Vect(k) is a ribbon category.

2. For $U_q(\mathfrak{sl}_2)$-mod, we again have to appeal to the extra struture that $U_q(\mathfrak{sl}_2)$ has. In particular, $U_q(\mathfrak{sl}_2)$ is a Ribbon Hopf Algebra which means there exists a special element, called the ribbon element which allows us to define the twist isomorphisms.

Why do we call a braided tensor category with a twist "balanced"? Consider Definition 4.6, which required each object of the category to have a dual object. In fact what we asked for there was for each object to have a right dual. We could instead have asked for left duals.

Definition 4.11. *For an object V in a tensor categorey $\mathcal{C}$, the left dual of V is an object ${}^*V$ with morphisms*

$$i'_V : I \to^* V \otimes V$$
$$e'_V : V \otimes^* V \to I.$$

such that

$$(\mathrm{id}_V \otimes e_V)(i_V \otimes \mathrm{id}_V) = \mathrm{id}_V \quad \textit{and} \quad (e_V \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^*} \otimes i_V) = \mathrm{id}_{V^*}.$$

In a balanced category, we can take ${}^*V = V^*$ (hence the name balanced) and define

$$i'_V = (\mathrm{id}_{V^*} \otimes \theta_V) c_{V,V^*} \circ i_V$$
$$e'_V = e_V \circ c_{V,V^*}(\theta_V \otimes \mathrm{id}_{V^*}).$$

This solves one problem as we can now assign a morphism to the cap in Figure 8. However, this creates another problem as there is no reason for the morphism represented by Figure 8 to be the same as the identity morphism. In fact, they are not the same and we can see this if we draw the morphism i'_V out in full as shown in Figure 9. This tangle has a box labelled θ_V where as there is no such box on the identity map. However, we don't want to have to use the box to distinguish between the two tangles as boxes never appear on links and we need i'_V and e'_v to just be a cap and cup respectively. Therefore, we need a way to encode into our category

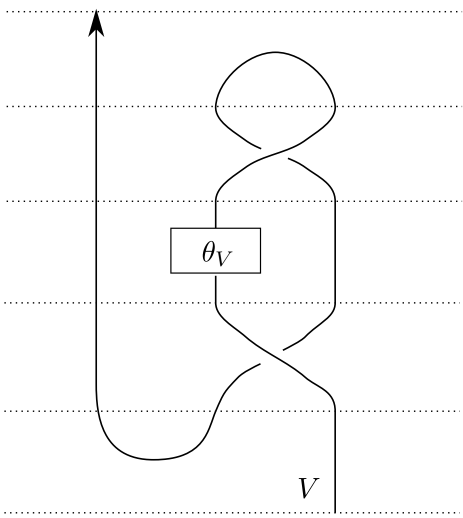


Figure 9: Using Reidemeister moves, this is isotopic to the straight line labelled with θ_V, which is not the same as the unlabelled line representing the identity.

of tangles that these two tangles are different. This invloves changing the category slightly by thickening the strands of all tangles slightly into ribbons. Figure 10 shows that the morphism from Figure 8 is now represented by a ribbon with a twist and the identity is a ribbon with no twist. This category is called the category of coloured ribbon tangles and we define it a bit

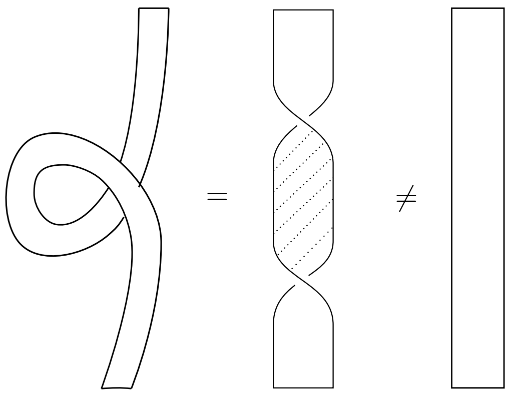


Figure 10: Thickening the strands into ribbons allows us to view a strand with a twist as different a strand with no twist.

more concretely now.

Definition 4.12. *Given a category $\mathcal{C}$, we define a category* $\text{Rib}_{\mathcal{C}}$ *to have*

- *Objects are words made up of objects of $\mathcal{C}$, where each object in the word is assigned either an up or down arrow;*
- *Morphisms are ribbons connecting two words, such that both ends of a ribbon must be attached to the same object of $\mathcal{C}$ with the orientation consistent along the ribbon.*

Composition is done by placing tangles on top of each other.

Now, taking $\mathcal{C} = U_q(\mathfrak{sl}_2) - \text{mod}$, we can define a functor $F : \text{Rib}_{\mathcal{C}} \to \mathcal{C}$. This maps objects and morphisms as in Figure 11.

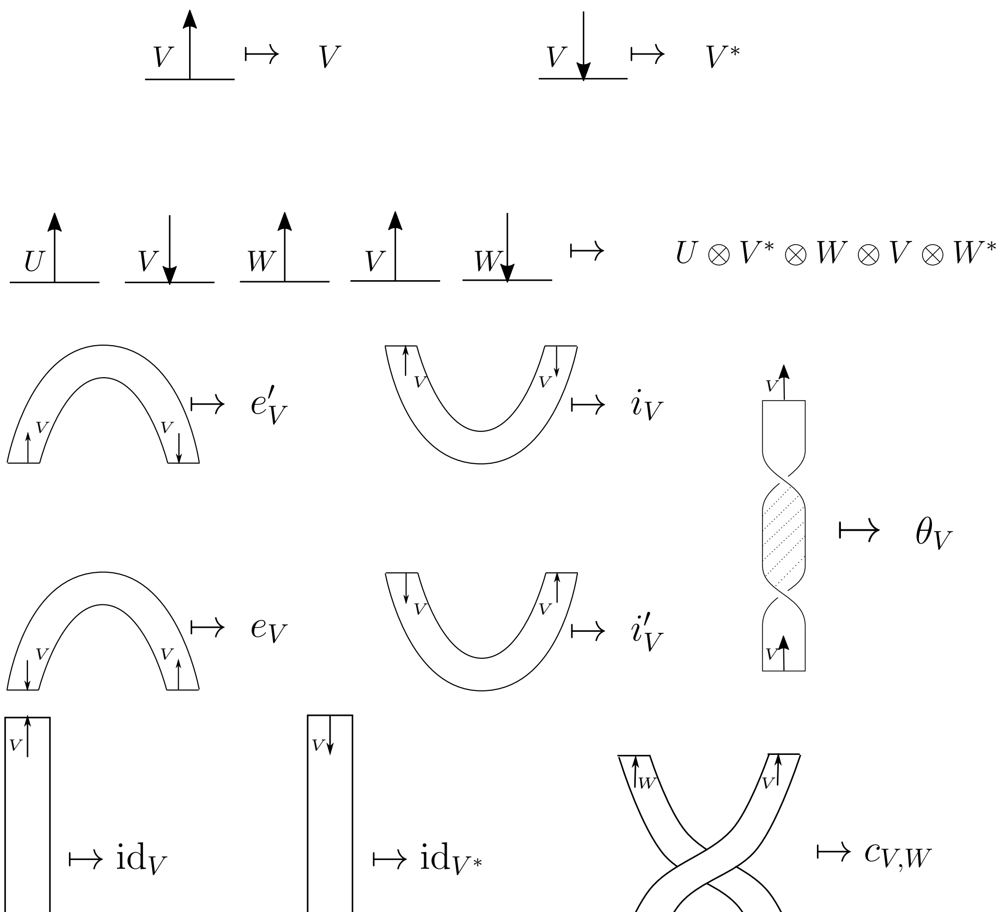
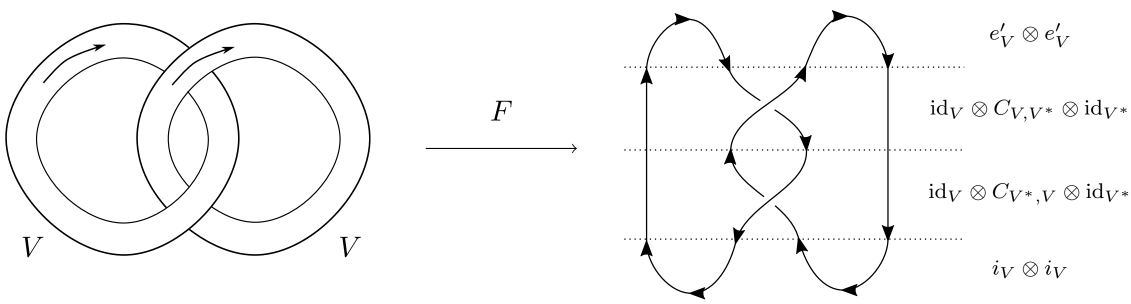


Figure 11: Shows where the functor $F : \text{Rib}_{\mathcal{C}} \to \mathcal{C}$ sends specific objects and morphisms.

Example 4.13. The following framed directed, coloured link maps as follows:



This morphism $(e'_V \otimes e'_V) \circ (\mathrm{id}_V \otimes C_{V,V^*} \otimes \mathrm{id}_{V^*}) \circ (\mathrm{id}_V \otimes C_{V^*,V} \otimes \mathrm{id}_{V^*}) \circ (i_V \otimes i_V)$ is a morphism from $\mathbb{C}(q) \to \mathbb{C}(q)$ and so can be thought of as an element of $\mathbb{C}(q)$. It is this element which we assign as the invariant to the link on the left hand side. However, we see there are several problems with our construction so far. The first is that this assignment required a choice: we chose to label both parts of the link by the representation V. If we had labelled them with a different choice, we would have got a different invariant. The second problem is that our invariant is only an invariant of framed, directed links rather than just links.

Unfortunately, we are never going to get rid of the choice involved. However, if we pick a represention V and let $\mathrm{Rib}_{\mathcal{C}}^V$ be the full subcategory of $\mathrm{Rib}_{\mathcal{C}}$ consisting of objects where every term is labelled by V, then we can view any framed, directed link, L as a morphism in this category, just by labelling every strand by V. Then, if F^V is the restriction of F to this subcategory, $F^V(L)$ is the link invariant assigned to L. Moreover, if we choose $\mathcal{C} = U_q(\mathfrak{sl}_2)$-mod and $V = \mathbb{C}^2$, the irreducible representation of dimension two, then we get the added structure that $V^* \cong V$ and so the direction does not matter. Hence, by making this choice, we get an invariant of framed links.

Finally, we address how we can get a link invariant from this. One way to go about this is to assign a rule that, for each link, tells us how to frame it and then we can use our invariant of framed links. One such rule is called blackboard framing. Informally, this just involves taking the link diagram and thickening the lines as they are without adding any twists. Examples are given in Figure 12. However, a problem occurs because the two knots in Figure 12 are isotopic

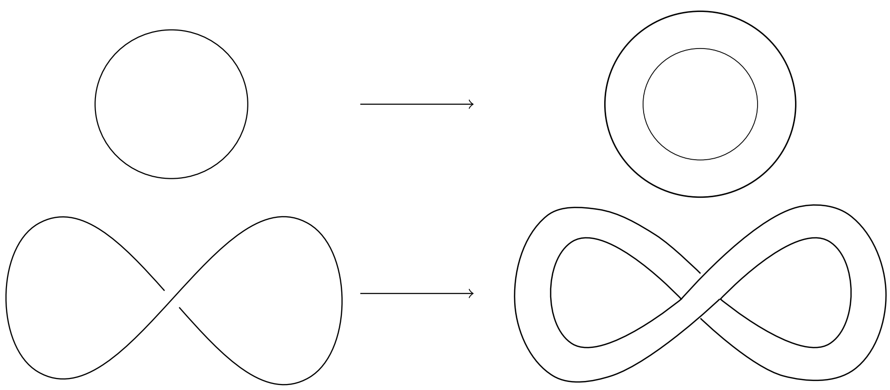
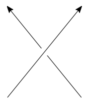
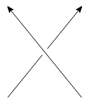


Figure 12: Examples of Blackboard Framing

and so should be given the same link invariant but the blackboard framing takes them to two different framed links: if we unfold the second it would be a circle with a twist rather than just a circle. To solve this, we use the writhe of the link diagram which keeps track of the crossings in a link diagram.

Definition 4.14. *In a link diagram L, we define positive and negative crossings as follows:*

$+ :=$  $- :=$ 

Then the writhe of a link diagram L, denoted $\omega(L)$ is the total number of crossings minus the total number of negative crossings.

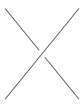
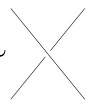
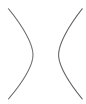
For example, the writhes of the two links in Figure 12 are 0 and 1 respectively. As you might expect, since the braiding θ_V keeps track of the twists in the correspondence we developed, we are going to use this to solve our problem with blackboard framing. Since V was chosen to be the irreducible representation of dimension 2, θ_V acts as a scalar and so we can think of it as an element of $\mathbb{C}(q)$. Then, if for a link L, we denote by L^b the blackboard framed link associated to L, the link invariant of L is

$$\theta_V^{\omega(L)} F^V(L^b).$$

Thus, we have constructed a knot invariant from the category $U_q(\mathfrak{sl}_2)$-mod using the large amount of structure it has. As a last remark, we note that the Jones Polynomial can be recovered from this construction. Recall that the Skein relation for the Jones polynomial was

$$t^{-1}P_{L_+}(t) - tP_{L_-}(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})P_{L_0}(t)$$

where L_+, L_- and L_0 were identical link diagrams except at a single crossing point where

$L_+ \sim$  $L_- \sim$  $L_0 \sim$ 

If we label each of the strands with our chosen representation V, then the three crossing points correspond to the maps $c_{V,V}, c_{V,V}^{-1}$ and $\mathrm{id}_{V\otimes V}$. However, we had a matrix for $c_{V,V}$ and the matrix for the identity is just the identity matrix. A quick calculation shows that

$$q^{-2}2c_{V,V} - q^2 c_{V,V}^{-1} = (q - q^{-1})\mathrm{id}_{V\otimes V}$$

and so taking $t = q^2$ we have precisely got the Skein relation back.

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