Higher Categories, Complete Segal Spaces and the Cobordism Hypothesis

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Contents

A note on notation I try to denote usual categories with italic letters, and higher categories (2-categories, (∞, n)-categories, etc.) with calligraphic letters.

1 Extending Cob(n)

1.1 Extending down

We've seen that the cobordism categories **Cob**(n) should really have more structure than just categories. In particular we should have, for every integer $k \leq n$, a k-category[1] $\mathbf{Cob}_k(n)$ with

$$\begin{aligned}
\text{objects} &\longleftrightarrow \text{closed oriented } (n-k)\text{-manifolds} \\
\text{1-morphisms} &\longleftrightarrow \text{oriented cobordisms} \\
\text{2-morphisms} &\longleftrightarrow \text{cobordisms between cobordisms} \\
&\vdots \\
k\text{-morphisms} &\longleftrightarrow \text{(diffeomorphism classes of) } n\text{-manifolds with corners}
\end{aligned}$$

We'd like to have a nice definition of k-category that includes $\mathbf{Cob}_k(n)$. Here's an obvious definition to make:

Definition 1.1.1. A **strict** 1**-category** is a category. A **strict** k**-category** is defined inductively as a category enriched over strict $(k-1)$-categories.

This definition is not the correct one. In particular $\mathbf{Cob}_k(n)$ is not a strict k-category since composition is not strictly associative, only associative up to isomorphism. We could adjust the definition of $\mathbf{Cob}_k(n)$ so that composition does become strictly associative, but this quickly gets messy.

Moral 1.1.2. We're going to need a better notion of k-category, where composition need only be associative up to isomorphism.

1.2 Extending up

Let's suppose we have a good definition of what a k-category is. Then we can define a (k,n)-category to be a k-category where all of the i-morphisms are invertible for $n < i \leq k$.

Example 1.2.1. A $(1,0)$-category should just be a groupoid.

It's also often useful to allow $k = \infty$; in fact we're going to define (∞, n)-categories later. This gives us a definition of (k,n)-categories simply by ignoring the morphisms above level k.

Example 1.2.2. An $(\infty, 0)$-category is an ∞**-groupoid**. Given a topological space X, we can form an ∞-groupoid $\pi_{\leq\infty}(X)$ called the **fundamental** ∞**-**

[1] In this section we'll treat higher categories at an informal level. Note that the concept of a "k-category" has not been defined!

groupoid of X, with

$$\begin{aligned} \text{objects} &\longleftrightarrow \text{points of } X \\ \text{1-morphisms} &\longleftrightarrow \text{paths between points} \\ \text{2-morphisms} &\longleftrightarrow \text{homotopies between paths} \\ \text{3-morphisms} &\longleftrightarrow \text{homotopies between homotopies} \\ &\vdots \end{aligned}$$

The fundamental groupoid of X remembers all of X up to weak homotopy equivalence. More formally, the fundamental groupoid construction is an equivalence between topological spaces (up to weak homotopy equivalence) and ∞-groupoids (up to equivalence). This assertion is known as the **homotopy hypothesis**[2]. This allows us to think of $(\infty, 0)$-categories as really being topological spaces. So as well as generalising category theory, higher category theory should also generalise topology.

Recall that in defining $\mathbf{Cob}(n)$, we defined a morphism $M \to N$ to be a diffeomorphism class of (oriented) cobordisms $M \to N$. Instead of considering two diffeomorphic cobordisms to be the same map, we could say that they differ by an invertible 2-morphism. Hence we should have an $(\infty, 1)$-category $\mathbf{Cob}^t(n)$ with

$$\begin{aligned} \text{objects} &\longleftrightarrow \text{closed oriented } (n-1)\text{-manifolds} \\ \text{1-morphisms} &\longleftrightarrow \text{oriented cobordisms} \\ \text{2-morphisms} &\longleftrightarrow \text{diffeomorphisms between cobordisms} \\ \text{3-morphisms} &\longleftrightarrow \text{isotopies between diffeomorphisms} \\ &\vdots \end{aligned}$$

Note that this definition allows us to keep track of the diffeomorphism groups of our cobordisms.

We can combine our two higher-categorical versions of $\mathbf{Cob}(n)$ into a single (∞, n)-category $\mathbf{Bord}_n$ with

[2] This is not a theorem yet, since we don't have a definition of ∞-groupoid. We could either **define** an ∞-groupoid to be a topological space, or we could regard the homotopy hypothesis as being a condition that our models of higher categories need to satisfy.

$$\begin{aligned}
\text{objects} &\longleftrightarrow \text{0-manifolds}\\
\text{1-morphisms} &\longleftrightarrow \text{cobordisms between 0-manifolds}\\
\text{2-morphisms} &\longleftrightarrow \text{cobordisms between cobordisms}\\
&\vdots\\
n\text{-morphisms} &\longleftrightarrow n\text{-manifolds with corners}\\
(n+1)\text{-morphisms} &\longleftrightarrow \text{diffeomorphisms}\\
(n+2)\text{-morphisms} &\longleftrightarrow \text{isotopies between diffeomorphisms}\\
&\vdots
\end{aligned}$$

Moral 1.2.3. We're going to need a good definition of (∞, n)-categories. Note that the disjoint union operation on 0-manifolds should turn $\mathbf{Bord}_n$into a symmetric monoidal (∞, n)-category.

1.3 Intuitive statement of the cobordism hypothesis

The cobordism hypothesis is stated in terms of **framed** cobordisms. This is a technical point and won't really concern us. Denote the (∞, n)-category of framed cobordisms by $\mathbf{Bord}_n^{\mathrm{fr}}$.

If $\mathcal{C}$ is a symmetric monoidal (∞, n)-category then consider the category of $\mathcal{C}$-valued fully extended framed TFTs: we can identify this category with the category $\mathbf{Fun}^{\otimes}(\mathbf{Bord}_n^{\mathrm{fr}}, \mathcal{C})$ of symmetric monoidal functors from $\mathbf{Bord}_n^{\mathrm{fr}}$ to $\mathcal{C}$.

The cobordism hypothesis more or less says that the evaluation functor $Z \mapsto Z(*)$ determines a bijection between isomorphism classes of $\mathcal{C}$-valued fully extended framed TFTs and isomorphism classes of objects in $\mathcal{C}$ satisfying suitable finiteness conditions.[3]

Remark 1.3.1. This specialises to a statement about $\mathbf{Cob}(n)$ by taking homotopy n-categories.

2 $(\infty, 1)$-categories as complete Segal spaces

We'll first define $(\infty, 1)$-categories and then soup up our definition in § 3 to get to (∞, n)-categories.

[3] By 'suitable finiteness conditions' we mean **full dualisability**, which we'll see a definition of in § 5. In $\mathbf{Vect}_k$ the fully dualisable objects are exactly the finite-dimensional vector spaces.

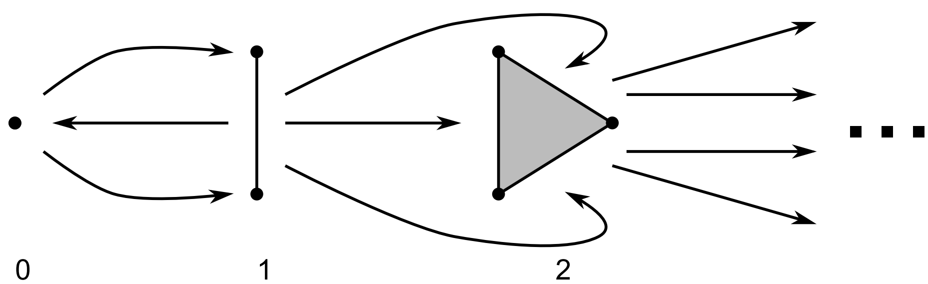
Intuitively, an $(\infty, 1)$-category should be a **topological category**; one where the hom-sets have the structure of topological spaces.[4] Higher morphisms are homotopies, homotopies between homotopies, and so on. However, while intuitive, this definition is very difficult to work with.

There are several other models[5] but we're going to use **complete Segal spaces** as our models for $(\infty, 1)$-categories since they generalise easily to (∞, n)-categories.

2.1 Preliminary: simplicial sets

Definition 2.1.1. The **simplex category** Δ has objects $[n] = \{0, 1, \ldots, n\}$ and morphisms the weakly order-preserving maps.

It looks like



where we've omitted the maps from [2] to [1]. The maps going to the right are the **face maps** and the maps going to the left are the **degeneracy maps**.

Definition 2.1.2. A **simplicial object** in a category C is a functor $\Delta^{\mathrm{op}} \to C$. More concretely a simplicial object is a collection of objects X_n indexed by the positive integers together with various face and degeneracy maps.

Definition 2.1.3. A **morphism** between two simplicial objects $F : \Delta^{\mathrm{op}} \to C$ and $G : \Delta^{\mathrm{op}} \to C$ is a natural transformation $F \to G$. Concretely, a morphism of simplicial objects is a collection of maps $X_n \to Y_n$ commuting with the face and degeneracy maps.

Proposition 2.1.4. *The collection of simplicial objects in a category C and their morphisms itself forms a category, which we denote sC.*

A simplicial object $X_\bullet$ looks like

$$X_0 \leftrightarrows X_1 \leftrightarrows X_2 \rightrightarrows \cdots$$

[4]One way to think about this is that an $(\infty, 1)$-category should be enriched in $(\infty, 0)$-categories, which are the same thing as topological spaces.

[5]A good account of these are given in [8].

We'll be interested in **simplicial sets**; simplicial objects in **Set**.[6] Later we'll be interested in simplicial topological spaces.

Example 2.1.5. Given a topological space X, we can define (functorially) a simplicial set $\mathrm{Sing}(X)$ that at level n is the set $\mathrm{Hom}(\Delta^n, X)$ of maps from the n-simplex $\Delta^n \subseteq \mathbb{R}^{n+1}$ to X. We also have a geometric realisation functor $|\cdot| : s\mathbf{Set} \to \mathbf{Top}$ and in fact $|\mathrm{Sing}(X)|$ is weakly homotopy equivalent to X. Simplicial sets are good combinatorial models of topological spaces.[7]

Example 2.1.6. Given a category C, the **nerve** is a simplicial set $N(C)$ which at level n consists of the strings $C_0 \xrightarrow{f_1} C_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} C_n$ of n composable morphisms. It's possible to recover C up to isomorphism from its nerve $N(C)$.

We might wonder what simplicial sets are the nerves of categories.

Proposition 2.1.7 (the Nerve Theorem)**.** *A simplicial set X is isomorphic to the nerve of a category if and only if for all $m, n \geq 0$ the diagram*

$$\begin{array}{ccc} X_{m+n} & \longrightarrow & X_m \\ \downarrow & & \downarrow \\ X_n & \longrightarrow & X_0 \end{array}$$

induced by the maps

$$\begin{array}{ccccc} & & 0 < 1 < \cdots < m & \longleftarrow\!\shortmid & 0 < 1 < \cdots < m \\ & & & & \\ m < m+1 < \cdots < m+n & & [m+n] & \longleftarrow & [m] & & m \\ \upmapsto & & \uparrow & & \uparrow & & \upmapsto \\ 0 < 1 < \cdots < n & & [n] & \longleftarrow & [0] & & 0 \\ & & & & \\ & & 0 & \longleftarrow\!\shortmid & 0 \end{array}$$

is Cartesian (i.e. a pullback diagram).

Whenever this diagram appears, we will fix the convention that the maps featuring are the maps described above.

2.2 Homotopy theory

Our philosophy is that $(\infty, 1)$-category theory should be category theory, but done in a homotopy-theoretic manner. This is because an $(\infty, 1)$-category is just a topological category where the higher morphisms are given by homotopies.

[6] A simplicial set is the same thing as a presheaf on Δ.

[7] Technically $s\mathbf{Set}$ and $\mathbf{Top}$ are Quillen equivalent (via these two functors), so they have the same homotopy theory.

Since Proposition 2.1.7 tells us that we can recover a category from its nerve, we'll try to code up the concept of a nerve in homotopy theory. We'll see that a Segal space is precisely this concept of 'homotopy nerve'. However we'll see that a Segal space alone won't quite be enough to recover an $(\infty, 1)$-category: we'll need some more conditions.

Definition 2.2.1. Let $X \xrightarrow{f} Z \xleftarrow{g} Y$ be a diagram of topological spaces[8] and continuous maps. The **homotopy fibre product** $X \times_Z^h Y$ of X and Y along f and g is the space $X \times_Z Z^{[0,1]} \times_Z Y$ whose points are triples (x, y, p) with $x \in X$, $y \in Y$ and $p : [0, 1] \to Z$ a path in Z from $f(x)$ to $g(y)$.

Remark 2.2.2. There is a canonical map from $X \times_Z Y$ to $X \times_Z^h Y$ given by $(x, y) \mapsto (x, y, p)$ where p is the constant path from $f(x) = g(y)$ to itself.

Example 2.2.3. Let X be a space and $p : * \to X$ be the inclusion of a basepoint. Then the homotopy fibre product of $* \xrightarrow{p} X \xleftarrow{p} *$ is the space ΩX of loops in X based at p. The usual fibre product is the one point space $*$.

Example 2.2.4. The homotopy fibre product of $* \xrightarrow{p} Y \xleftarrow{f} X$ is the homotopy fibre of f over the basepoint p in Y.

The usual fibre product of topological spaces does not respect homotopy equivalences. The homotopy fibre product is invariant under homotopy equivalence: if we replace f and g by homotopic maps then the weak homotopy type of $X \times_Z^h Y$ does not change.

Remark 2.2.5. Another nice property of the homotopy fibre product is that we have a long exact sequence of homotopy groups

$$\cdots \to \pi_n(X \times_Z^h Y) \to \pi_n(X) \times \pi_n(Y) \to \pi_n(Z) \to \cdots \to \pi_0(X) \times \pi_0(Y)$$

Proposition 2.2.6. *The homotopy fibre product $X \times_Z^h Y$ comes with two canonical projection maps to X and Y making the diagram*

$$\begin{array}{ccc} X \times_Z^h Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

commute up to canonical homotopy. Moreover if the square

$$\begin{array}{ccc} W & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

*is homotopy commutative then there is a unique up to homotopy map $W \to X \times_Z^h Y$ making the two triangles obtained strictly commutative. For this reason we often call $X \times_Z^h Y$ the **homotopy pullback** of X and Y along f and g.*

[8]For technical reasons we need to work with a 'convenient category' of spaces. For example we can use CGH (compactly generated Hausdorff) spaces as in [5] or CGWH (compactly generated weak Hausdorff) spaces as in [6]. For concreteness we may suppose that all topological spaces are CGWH.

Definition 2.2.7. A homotopy commutative square

$$\begin{array}{ccc} W & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

is **homotopy Cartesian** (or just **h-Cartesian**) if there is a weak homotopy equivalence $W \to X \times_Z^h Y$ such that the triangles obtained are strictly commutative.

Neither

$$\text{h-Cartesian} \implies \text{Cartesian}$$

nor

$$\text{Cartesian} \implies \text{h-Cartesian}$$

is true in general! If our maps are sufficiently nice (e.g. if $X \to Z$ is a fibration) then a Cartesian square is homotopy Cartesian. In this situation we can compute the homotopy fibre product by computing the usual fibre product.

2.3 Segal spaces

Definition 2.3.1 (Rezk)**.** A simplicial topological space $X_\bullet$ is a **Segal space** if for all $m, n \geq 0$ the diagram

$$\begin{array}{ccc} X_{m+n} & \longrightarrow & X_m \\ \downarrow & & \downarrow \\ X_n & \longrightarrow & X_0 \end{array}$$

is h-Cartesian. We can equivalently specify that for all n the **Segal maps**

$$X_n \to \underbrace{X_1 \times_{X_0}^h X_1 \times_{X_0}^h \cdots \times_{X_0}^h X_1}_{n}$$

are weak homotopy equivalences.

Remark 2.3.2. This is not a universally accepted definition. Some authors, for example [2], specify in addition that $X_\bullet$ should be **Reedy fibrant**, a 'niceness' condition on simplicial spaces that ensures that the homotopy pullback $X_m \times_{X_0}^h X_n$ is the usual pullback $X_m \times_{X_0} X_n$. In this case it's enough to demand that $X_{m+n} \to X_m \times_{X_0} X_n$ is a weak homotopy equivalence. Reedy fibrancy is a technical condition that can always be satisfied. For more on model category theory and the definition of Reedy fibrancy, the reader can consult e.g. Appendix A.2 of [7].

What do Segal spaces have to do with $(\infty,1)$-categories? Let's suppose for the moment that we already have a good theory of $(\infty,1)$-categories. Just like a 1-category has an underlying groupoid, obtained by throwing away all of the noninvertible morphisms, an $(\infty,1)$-category should have an underlying ∞-groupoid obtained in the same way:

Idea 2.3.3. Let $\mathcal{C}$ be any $(\infty,1)$-category. We can loosely define the **underlying ∞-groupoid** of $\mathcal{C}$, which I will denote $\pi_{\leq\infty}(\mathcal{C})$, to be the ∞-groupoid with

$$\begin{aligned} \text{objects} &\longleftrightarrow \text{objects of } \mathcal{C} \\ \text{1-morphisms} &\longleftrightarrow \text{invertible 1-morphisms in } \mathcal{C} \\ \text{2-morphisms} &\longleftrightarrow \text{2-morphisms between invertible 1-morphisms of } \mathcal{C} \\ &\vdots \end{aligned}$$

Since we can identify ∞-groupoids with topological spaces, we may think of $\pi_{\leq\infty}(\mathcal{C})$ as a topological space $B_0\mathcal{C}$ which we refer to as the **classifying space for objects of** $\mathcal{C}$. Note that by definition the fundamental ∞-groupoid of $B_0\mathcal{C}$ is the underlying ∞-groupoid of $\mathcal{C}$.

Clearly $B_0\mathcal{C}$ should not in general encode all of the information about $\mathcal{C}$. For example it doesn't know about noninvertible morphisms or how composition works. We can extend the above definition to get classifying spaces for n-morphisms of $\mathcal{C}$ (since we can think of an object as a 0-morphism), and hopefully this collection of classifying spaces should allow us to recover $\mathcal{C}$.

Idea 2.3.4. Let $[n]$ be the 1-category associated to the ordered set $\{0,1,2\ldots,n\}$. Let $\mathcal{C}$ be an $(\infty,1)$-category. We can think of an n-morphism in $\mathcal{C}$ as a functor $[n]\to\mathcal{C}$. The collection $\mathrm{Fun}([n],\mathcal{C})$ of functors $[n]\to\mathcal{C}$ itself naturally has the structure of an $(\infty,1)$-category, so it has an underlying ∞-groupoid $\pi_{\leq\infty}(\mathrm{Fun}([n],\mathcal{C}))$. Let $B_n\mathcal{C}$ be the topological space associated to this ∞-groupoid. We call $B_n\mathcal{C}$ the **classifying space for n-morphisms in** $\mathcal{C}$. Again, by definition the fundamental ∞-groupoid of $B_n\mathcal{C}$ is the underlying ∞-groupoid of $\mathrm{Fun}([n],\mathcal{C})$.

What kind of object should the collection $B_\bullet\mathcal{C}$ be? Moreover, to what extent does it determine $\mathcal{C}$? The answer to the first question is that $B_\bullet\mathcal{C}$ should be a simplicial space, and moreover a Segal space. The Segal conditions formalise the idea that giving a chain

$$A_0\to A_1\to\cdots\to A_{n+m}$$

of composable morphisms should be equivalent to giving two chains

$$A_0\to\cdots\to A_n \qquad A_n\to\cdots\to A_{n+m}$$

and moreover that it should not matter where we break the chains. To answer the second question, we can try to define an 'inverse' to the operation $\mathcal{C}\to B_\bullet\mathcal{C}$ and see if we need to add any extra data to a general Segal space in order to extract an ∞-category.

Idea 2.3.5. Given a Segal space $X_\bullet$ we should be able to construct an $(\infty,1)$-category $\mathcal{C}(X_\bullet)$ which has

$$\begin{aligned} \text{objects} &\longleftrightarrow \text{points of } X_0 \\ \text{Mapping spaces } \mathrm{Map}(x,y) &\longleftrightarrow \{x\} \times^h_{X_0} X_1 \times^h_{x_0} \{y\} \\ \text{composition law} &\longleftrightarrow \text{given by } X_2 \\ \text{higher associativity information} &\longleftrightarrow \text{given by } X_3, X_4 \dots \end{aligned}$$

Observe that the connected components of the space $\mathrm{Map}(x,y)$ should be precisely the homotopy classes of 1-morphisms in $\mathcal{C}(X_\bullet)$. With this in mind, we can construct a 1-category from a Segal space:

Definition 2.3.6. The **homotopy category** $hX_\bullet$ of a Segal space $X_\bullet$ is the category whose objects are the points of X_0, and whose homsets are

$$\begin{aligned} \mathrm{Hom}_{hX_\bullet}(x,y) &:= \pi_0(\mathrm{Map}(x,y)) \\ &= \pi_0(\{x\} \times^h_{X_0} X_1 \times^h_{x_0} \{y\}) \end{aligned}$$

Remark 2.3.7. The homotopy category of $X_\bullet$ records some of the basic information about $\mathcal{C}(X_\bullet)$ - it knows what the objects should be, for example - but it forgets all of the homotopical information by identifying all homotopic maps. It can be thought of as a 1-categorical 'flattening' of the $(\infty,1)$-category $\mathcal{C}(X_\bullet)$.

2.4 Completeness

If we start with a general Segal space $X_\bullet$, how does it compare to the Segal space $Y_\bullet := B_\bullet(\mathcal{C}(X_\bullet))$? The fundamental groupoid of Y_0 is the classifying space for 0-morphisms of $\mathcal{C}(X_\bullet)$. This receives a map from the fundamental groupoid of X_0 but this map is not necessarily an equivalence, since there may be invertible 1-morphisms in $\mathcal{C}(X_\bullet)$ which do not come from paths in X_0. We'd like to impose an extra condition on our Segal spaces which ensures that every invertible 1-morphism in $\mathcal{C}(X_\bullet)$ comes from an essentially unique path in X_0.

Definition 2.4.1. Let p_i be the map from $[0]$ to $[1]$ given by $0 \mapsto i$. For any Segal space $X_\bullet$ write $p_i^* : X_1 \to X_0$ for the map induced by p_i. If $f \in X_1$ then write $x := p_0^*(f)$ and $y := p_1^*(f)$ so that we can think of f as a path from x to y. The map $\{f\} \to \{x\} \times_{X_0} X_1 \times_{x_0} \{y\} \to \{x\} \times^h_{X_0} X_1 \times^h_{x_0} \{y\}$ determines an element $[f]$ of $\mathrm{Hom}_{hX_\bullet}(x,y) = \pi_0(\{x\} \times^h_{X_0} X_1 \times^h_{x_0} \{y\})$. Say that f is **invertible** if $[f]$ is an isomorphism.

Example 2.4.2. If $X_\bullet$ is a Segal space let $\delta : X_0 \to X_1$ be the map induced by the unique map $[1] \to [0]$. Then for every $x \in X_0$, the map $[\delta(x)]$ is the identity map id_x in the homotopy category. So $\delta(x)$ is invertible for every x.

Definition 2.4.3. If $Z \subseteq X_1$ is the subspace of invertible elements of a Segal space $X_\bullet$, then say that $X_\bullet$ is **complete** if $\delta : X_0 \to Z$ is a weak homotopy equivalence.

So a complete Segal space is one where every isomorphism in $\mathcal{C}(X_\bullet)$arises from an essentially unique path in X_0. In fact, if $X_\bullet$ is complete then we should have an equivalence $X_\bullet \cong B_\bullet(\mathcal{C}(X_\bullet))$.

Remark 2.4.4. If one is more careful and starts with a rigorous axiomatisation of $(\infty, 1)$-categories then the above assertions and intuitive ideas can be turned into theorems. This was done by Toën in [9].

We've seen that the well-defined theory of complete Segal spaces should correspond to the as-yet-undefined theory of $(\infty, 1)$-categories. With this in mind, we make the following rather bold definition:

Definition 2.4.5. An $(\infty, 1)$**-category** is a complete Segal space.

Proposition 2.4.6 (Rezk)**.** *Any Segal space $X_\bullet$ has a completion; i.e. admits a homotopy universal morphism*[9] $X_\bullet \to Y_\bullet$ *where $Y_\bullet$ is complete. In general $Y_\bullet$ is unique up to homotopy and we refer to it as the* ***completion*** *of $X_\bullet$, denoted $\hat{X}_\bullet$. The map $X_\bullet \to \hat{X}_\bullet$ is functorial.*

Remark 2.4.7. Complete Segal spaces are the fibrant objects of a suitable model structure on the category of simplicial spaces, just as quasicategories are the fibrant objects of the Joyal model structure on the category of simplicial sets.

3 (∞, n)-categories as n-fold complete Segal spaces

Now we have a definition of $(\infty, 1)$-category as a certain functor $\Delta^{\mathrm{op}} \to \mathbf{Top}$, we're going to generalise this and define an (∞, n)-category as a certain functor $(\Delta^{\mathrm{op}})^{\times n} \to \mathbf{Top}$.

Definition 3.1.1. An n**-fold simplicial object** in a category C is a functor

$$\underbrace{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \times \cdots \times \Delta^{\mathrm{op}}}_{n} \to C$$

Example 3.1.2. A 0-fold simplicial object is an object. A 1-fold simplicial object is just a simplicial object in the usual sense.

In general an n-fold simplicial object in a category C is a collection $X_{i_1 \cdots i_n}$ of objects of C indexed by n-tuples of nonnegative integers $\underline{i} = (i_1, \ldots, i_n)$ along with a collection of face and degeneracy maps. We'll always use an underbar to denote multiindices in this manner. We think of n-fold simplicial objects as having n 'directions' in which to compose.

Definition 3.1.3. An n**-fold simplicial space** is an n-fold simplicial object in the category $\mathbf{Top}$.

[9] A morphism of Segal spaces is a morphism of the underlying simplicial spaces.

Definition 3.1.4. A map $X \to Y$ of n-fold simplicial spaces is a **weak homotopy equivalence** if all of the maps $X_{\underline{i}} \to Y_{\underline{i}}$ are weak homotopy equivalences.

Definition 3.1.5. A diagram

$$\begin{array}{ccc} W & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

of n-fold simplicial spaces is **homotopy Cartesian** if for all multiindices $\underline{i}$ the square

$$\begin{array}{ccc} W_{\underline{i}} & \longrightarrow & Y_{\underline{i}} \\ \downarrow & & \downarrow \\ X_{\underline{i}} & \longrightarrow & Z_{\underline{i}} \end{array}$$

is homotopy Cartesian.

Definition 3.1.6. An n-fold simplicial space X is **essentially constant** if it's weakly homotopy equivalent to a constant n-fold simplicial space.

Via currying, whenever $n > 0$ we can always think of an n-fold simplicial object in C as a simplicial object in the category of $(n-1)$-fold simplicial objects in C. This idea will form the basis of our inductive definition of an n-fold complete Segal space.

Definition 3.1.7. For $n > 0$ an n-fold simplicial space X, thought of as a simplicial object in the category of $(n-1)$-fold simplicial spaces, is said to be an **n-fold Segal space** if the following conditions are met:

i) Every X_k is an $(n-1)$-fold Segal space.

ii) For all m and l the diagram

$$\begin{array}{ccc} X_{m+l} & \longrightarrow & X_m \\ \downarrow & & \downarrow \\ X_l & \longrightarrow & X_0 \end{array}$$

is a homotopy Cartesian square of $(n-1)$-fold simplicial spaces.

iii) X_0 is an essentially constant $(n-1)$-fold simplicial space.

Moreover, we say that an n-fold Segal space is **complete** if

iv) Each X_k is a complete $(n-1)$-fold Segal space.

v) The Segal space $Y_\bullet = X_{\bullet,0,0,\ldots,0}$ is complete.

Definition 3.1.8. An (∞, n)**-category** is a complete n-fold Segal space.

Proposition 3.1.9. *Any n-fold Segal space has a completion.*

Loosely, an n-fold complete Segal space is a 'fattened' or 'spread out' version of an (∞, n)-category. Some illuminating diagrams are given in §2.2.1 of [2].

4 The (∞, n)-category $\mathbf{Bord}_n$

In this section we'll code up our ideas about $\mathbf{Bord}_n$ to define an n-fold simplicial space $\mathbf{PBord}_n$. We'll indicate how this is an n-fold Segal space that in general is not complete. Then we can define the (∞, n)-category $\mathbf{Bord}_n$ to be the completion $\widehat{\mathbf{PBord}}_n$ of $\mathbf{PBord}_n$. Our exposition will be fairly informal; for a more rigorous explanation see §2 of [2].

4.1 The level sets $\left(\mathbf{PBord}_n^V\right)_{\underline{k}}$

We want to think of $(\mathbf{PBord}_n)_{(k_1,\ldots,k_n)}$ as a collection of $k_1 k_2 \cdots k_n$ composed cobordisms, with k_i cobordisms in the i^{th} direction.

Idea 4.1.1. Cobordisms are easier to deal with when we consider them as submanifolds of some large $\mathbb{R}^m$. So we'll define sets of cobordisms living in $\mathbb{R}^m$ for varying m, and then take a limit over m. Whitney's embedding theorem will ensure that we get all of the cobordisms, since every l-dimensional manifold can be embedded in $\mathbb{R}^{2l}$.

Definition 4.1.2. Let V be a finite-dimensional real vector space and fix a multiindex $\underline{k} = (k_1, \ldots, k_n)$. Define $\left(\mathbf{PBord}_n^V\right)_{\underline{k}}$ to be the set of tuples

$$(M, (t_0^i, \ldots, t_{k_i}^i)_{i=1\ldots n})$$

satisfying the following:

i) For each $1 \leq i \leq n$, $t_0^i \leq \cdots \leq t_{k_i}^i$ is an ordered tuple of $k_i + 1$ real numbers.

ii) M is a closed n-dimensional submanifold of $V \times \mathbb{R}^n$ and the composition $\pi : M \hookrightarrow V \times \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ is proper[10].

iii) For a subset S of $\{1, \ldots, n\}$ let $p_S : M \to \mathbb{R}^S$ denote the composition $M \xrightarrow{\pi} \mathbb{R}^n \twoheadrightarrow \mathbb{R}^S$. Then we require that for every $1 \leq i \leq n$ and every $0 \leq j \leq k_i$, that for all $x \in p_{\{i\}}^{-1}\left(t_j^i\right)$, the map $p_{\{i,\ldots,n\}}$ is submersive[11] at x.

Remark 4.1.3. What's the motivation behind this definition? If we want to think of M as being a collection of composed cobordisms, the numbers t_j^i record the 'cutting points' where we glue two cobordisms together. So the region of M between the hyperplanes corresponding to t_j^i and t_{j+1}^i should be the $(j+1)^{\text{st}}$ cobordism glued in the i^{th} direction.

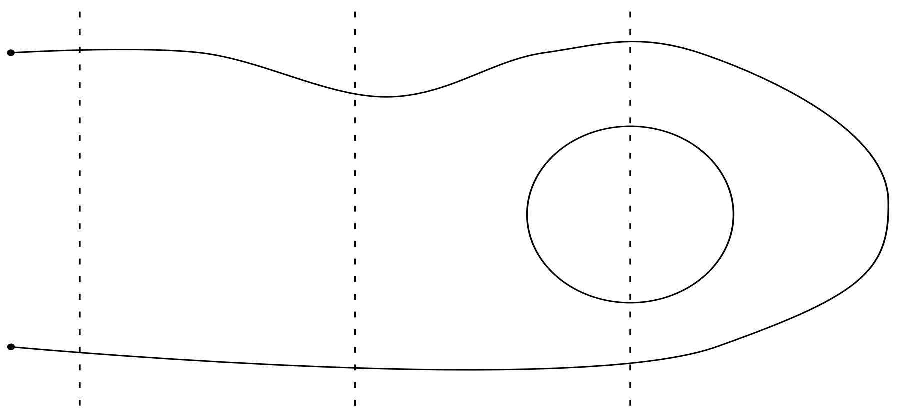
[10] A map is **proper** if preimages of compact sets are compact.

[11] A map $f : M \to N$ is **submersive at** $m \in M$ if the differential $\mathrm{d}f_x : T_x M \to T_x N$ is surjective. A map is **submersive** if it's submersive at every point of its domain.

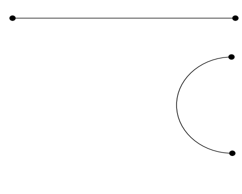
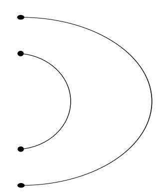
Condition iii) says that in particular the set $p_{\{n\}}^{-1}\left(t_j^n\right)$ is an $(n-1)$-dimensional submanifold that we can think of as one of the $(n-1)$-cobordisms that we glue together to get M.

Furthermore the set $p_{\{n-1,n\}}^{-1}\left\{t_{j_{n-1}}^{n-1}, t_{j_n}^n\right\}$ is an $(n-2)$-dimensional manifold that is one of the $(n-2)$-cobordisms joined by an $(n-1)$-cobordism. Similarly, the preimage $p_{\{m,\ldots,n\}}^{-1}\left\{t_{j_m}^m, \ldots, t_{j_n}^n\right\}$ is an $(m-1)$-dimensional manifold that we can loosely think of as one of our $(m-1)$-morphisms.

Example 4.1.4. Here is an element of $\mathbf{PBord}_1^{\mathbb{R}}$:



The cutting points indicated by the dotted lines allow us to view this as a composition of the three 1-cobordisms

 ,  and 

4.2 The topological spaces $\left(\mathbf{PBord}_n^V\right)_{\underline{k}}$

Fact 4.2.1 ([10], Chapter II). The set $\mathrm{Emb}(X, Y)$ of smooth embeddings of a smooth manifold X into a smooth manifold Y has a topology, the **Whitney C^∞ topology**.

Theorem 4.2.2 ([11])**.** *The space* $\mathrm{Sub}(V \times \mathbb{R}^n)$ *of closed n-dimensional submanifolds of $V \times \mathbb{R}^n$ can be identified with the space*

$$\bigsqcup_{[L]} \mathrm{Emb}(L, V \times \mathbb{R}^n)/\mathrm{Diff}(L)$$

where the disjoint union is taken over diffeomorphism classes of n-dimensional manifolds L. Moreover the topology on $\mathrm{Sub}(V \times \mathbb{R}^n)$ *has neighbourhood basis at* $M \subseteq V \times \mathbb{R}^n$ *the sets*

$$\{N \subseteq V \times \mathbb{R}^n : N \cap K = f(M) \cap K \text{ for all } f \in W\}$$

where K *is a compact subset of* $V \times \mathbb{R}^n$ *and* $W \subseteq \mathrm{Emb}(M, V \times \mathbb{R}^n)$ *is a neighbourhood (in the Whitney* C^∞ *topology) of the inclusion* $M \hookrightarrow V \times \mathbb{R}^n$*.*

Remark 4.2.3. The space $\mathrm{Sub}(V \times \mathbb{R}^n)$ is sometimes denoted by $\Psi(V \times \mathbb{R}^n)$.

Since we can view $\left(\mathbf{PBord}_n^V\right)_{\underline{k}}$ as a subset of $\mathrm{Sub}(V \times \mathbb{R}^n) \times \mathbb{R}^k$, for some $k \in \mathbb{N}$ depending only on $\underline{k}$, we can give $\left(\mathbf{PBord}_n^V\right)_{\underline{k}}$ the subspace topology.

4.3 The n-fold simplicial space $\mathbf{PBord}_n$

Proposition 4.3.1. *There are face and degeneracy maps making the collection of spaces*

$$\left\{\left(\mathbf{PBord}_n^V\right)_{\underline{k}} : \underline{k} \in \mathbb{N}^n\right\}$$

into an n-fold simplicial space.

Call this n-fold simplicial space $\mathbf{PBord}_n^V$. Loosely, the face maps forget a number t^i_j whereas the degeneracy maps repeat a number t^i_j.

Now we can remove the dependence on the vector space V. Let $\mathbb{R}^\infty$ be the unique real vector space of countably infinite dimension. We define the n-fold simplicial space $\mathbf{PBord}_n$ to be the limit

$$\mathbf{PBord}_n := \varinjlim_{V \subseteq \mathbb{R}^\infty} \mathbf{PBord}_n^V$$

4.4 The n-fold Segal spaces $\mathbf{PBord}_n$ and $\mathbf{Bord}_n$

We need to verify that the n-fold simplicial space $\mathbf{PBord}_n$is in fact an n-fold Segal space. The important point is to prove that the Segal map

$$\begin{array}{c} (\mathbf{PBord}_n)_{k_1,\dots,k_i+k_i',\dots,k_n} \\ \downarrow \\ (\mathbf{PBord}_n)_{k_1,\dots,k_i,\dots,k_n} \times^h_{(\mathbf{PBord}_n)_{k_1,\dots,0,\dots,k_n}} (\mathbf{PBord}_n)_{k_1,\dots,k_i',\dots,k_n} \end{array}$$

is a weak homotopy equivalence.

Fact 4.4.1. $\mathbf{PBord}_n$is nice enough for the homotopy fibre product above to be weakly homotopy equivalent to the usual fibre product. More technically, $\mathbf{PBord}_n$is Reedy fibrant - see 2.3.2 for further discussion.

Corollary 4.4.2. *To verify that the above Segal map is a weak homotopy equivalence, we may replace the homotopy fibre product with the genuine fibre product of topological spaces.*

Replacing the homotopy pullback with the usual pullback, we see that an element of

$$(\mathbf{PBord}_n)_{k_1,\ldots,k_i,\ldots,k_n} \underset{(\mathbf{PBord}_n)_{k_1,\ldots,0,\ldots,k_n}}{\times} (\mathbf{PBord}_n)_{k_1,\ldots,k_i',\ldots,k_n}$$

is a pair of submanifolds M and N of $V \times \mathbb{R}^n$ for some V, together with data allowing us to glue them on their intersection. Gluing them together gets us an element of $(\mathbf{PBord}_n)_{k_1,\ldots,k_i+k_i',\ldots,k_n}$. The Segal map is in fact a homeomorphism, not just a homotopy equivalence. We can now deduce the following:

Theorem 4.4.3. $\mathbf{PBord}_n$ *is an n-fold Segal space.*

The n-fold Segal space $\mathbf{PBord}_n$is not in general complete. We define $\mathbf{Bord}_n := \widehat{\mathbf{PBord}_n}$ to be its completion. Then $\mathbf{Bord}_n$is an (∞, n)-category.

Remark 4.4.4. The spaces $\mathbf{PBord}_1$ and $\mathbf{PBord}_2$ **are** complete. However, for $n \geq 6$, $\mathbf{PBord}_n$ is **not** complete; this is because not all invertible cobordisms $M \to N$ arise from diffeomorphisms $M \to N$. The **s-cobordism theorem** says that for $n \geq 6$, this statement is equivalent to the vanishing of an invariant of the cobordism known as the **Whitehead torsion**. It's known that for $n \geq 6$ that there are n-bordisms which have nontrivial Whitehead torsion, and hence that $\mathbf{PBord}_n$is not complete.

4.5 Extra structure on $\mathbf{Bord}_n$

Most importantly, $\mathbf{Bord}_n$is a **symmetric monoidal** (∞, n)**-category**, which means that it has a symmetric monoidal structure (given by the disjoint union) compatible with the (∞, n) structure.

We can also restrict to cobordisms with certain properties: for example there is an (∞, n)-category $\mathbf{Bord}_n^{\mathrm{fr}}$ of framed cobordisms, and an (∞, n)-category $\mathbf{Bord}_n^{\mathrm{or}}$ of oriented cobordisms. Both of these categories also carry a symmetric monoidal structure.

The (∞, n)-category $\mathbf{Bord}_n^{\mathrm{fr}}$ of framed cobordisms will be our focus from now on, since the Cobordism Hypothesis is stated in terms of framed cobordisms.

Note on the constructions The construction of $\mathbf{Bord}_n$ outlined above is similar to Lurie's definition in [1]. Lurie's original definition contained an error, and this was corrected by Calaque and Scheimbauer in [2] (which consists mainly of material from [3]) who construct their spaces differently.

They first construct a Segal space of intervals in $\mathbb{R}^n$ and then lift this Segal space structure to $\mathbf{PBord}_n$. The definitions in [2] correspond roughly to the definitions here by taking our t_j^i to be points in their intervals.

5 Adjoints and dualisablity

Given a topological field theory $Z : \mathbf{Bord}_n^{\mathrm{fr}} \to \mathcal{C}$, we'd like to classify the kind of objects of $\mathcal{C}$ that could be the image of the 0-manifold $*$ under Z. Such objects should satisfy some finiteness condition: for example when $n = 1$ and $\mathcal{C} = \mathbf{Vect}_k$ we saw that $Z(*)$ had to be finite-dimensional, and conversely that any finite-dimensional vector space can be obtained as the image of $*$ under some TFT.

The correct analogue of finite-dimensionality in the ∞-categorical setting is **full dualisability**, and to define this is the goal of the current section.

It turns out that requiring dualisability for objects is not enough: we'll also need a notion of dualisability for k-morphisms as well. In the 2-category **Cat** we already have a reasonable notion of dualisability for 1-morphisms: a left dual (if it exists) for a functor $F : C \to D$ should be its left adjoint $G : D \to C$. We extend this definition to general 2-categories and then to general (∞, n)-categories. Adjoints and duals are very closely related in higher categories.

All of this section is from [1].

5.1 Duals for objects

Recall the following 1-categorical definition:

Definition 5.1.1. Let C be a monoidal category. Let V be an object of C. A **right dual** for V is the data of an object $V^\vee$ and maps

$$\begin{aligned} \mathrm{ev} : V \otimes V^\vee \to 1 \qquad & \text{the } \textbf{evaluation map} \\ \mathrm{coev} : 1 \to V^\vee \otimes V \qquad & \text{the } \textbf{coevaluation map} \end{aligned}$$

such that the triangles[12]

$$\begin{array}{ccc} V & & \\ {\scriptstyle \mathrm{id}_V \otimes \mathrm{coev}}\downarrow & \searrow^{\mathrm{id}_V} & \\ V \otimes V^\vee \otimes V & \xrightarrow[\mathrm{ev}\otimes \mathrm{id}_V]{} & V \end{array} \qquad \begin{array}{ccc} V^\vee & & \\ {\scriptstyle \mathrm{coev}\otimes \mathrm{id}_{V^\vee}}\downarrow & \searrow^{\mathrm{id}_{V^\vee}} & \\ V^\vee \otimes V \otimes V^\vee & \xrightarrow[\mathrm{id}_{V^\vee}\otimes \mathrm{ev}]{} & V^\vee \end{array} \tag{1}$$

commute. We also say in this situation that V is a **left dual** of $V^\vee$.

Remark 5.1.2. If C is symmetric monoidal, then the notions of right dual and left dual coincide and we refer simply to the **dual**.

Example 5.1.3. If C is the symmetric monoidal category $\mathbf{Vect}_k$ (with monoidal structure given by the usual tensor product over k) then a vector space V has a dual if and only if it is finite-dimensional. More specifically, we can always define a space $V^* = \mathrm{Hom}(V, k)$ and an evaluation map $V \otimes_k V^* \to k$, but we can only define a compatible coevaluation map if V is finite-dimensional.

Proposition 5.1.4. *Left and right duals (if they exist) are unique up to unique isomorphism.*

We can easily extend the definition of a dualisable object to higher categories, by taking the homotopy category.

Definition 5.1.5. Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category. Say that an object X of $\mathcal{C}$ is **dualisable** if it admits a dual when considered as an object of the homotopy category $h\mathcal{C}$.

If Z is an oriented or framed topological field theory with target $\mathcal{C}$, then any object X of $\mathcal{C}$ with $X = Z(*)$ must be dualisable since we can obtain $X^\vee$ by evaluating Z on a point with the opposite orientation to that of $*$. In general the condition that X be dualisable is not strong enough for such a TFT to exist. However, for $n = 1$ it turns out that dualisability is sufficient, so this problem will only manifest itself in higher dimensions.

In general we should require that the morphisms in $\mathcal{C}$ should also have duals, which leads us to the notion of adjoints.

5.2 Adjoints in 2-categories

Recall the unit-counit definition of an adjunction:

Definition 5.2.1. Let C, D be two categories and $F : C \to D$ and $G : D \to C$ two functors. An **adjunction** between F and G consists of two natural

[12]These triangles are usually presented as pentagons; here we have ignored the associators and the isomorphisms $X \otimes 1 \xrightarrow{\sim} X \xleftarrow{\sim} 1 \otimes X$.

transformations

$$\begin{aligned} u &: \mathrm{id}_C \Rightarrow GF && \text{the } \textbf{unit} \\ v &: FG \Rightarrow \mathrm{id}_D && \text{the } \textbf{counit} \end{aligned}$$

such that the following two triangles[13] commute:

$$\begin{array}{ccc} F \circ \mathrm{id}_C & & \\ {\scriptstyle \mathrm{id}_F \times u}\downarrow & \searrow^{\mathrm{id}_F} & \\ F \circ G \circ F & \xrightarrow[v \times \mathrm{id}_F]{} & \mathrm{id}_D \circ F \end{array} \qquad \begin{array}{ccc} \mathrm{id}_C \circ G & & \\ {\scriptstyle u \times \mathrm{id}_G}\downarrow & \searrow^{\mathrm{id}_G} & \\ G \circ F \circ G & \xrightarrow[\mathrm{id}_G \times v]{} & G \circ \mathrm{id}_D \end{array} \tag{2}$$

In this situation we say that F is a **left adjoint** of G and that G is a **right adjoint** of F.

Note that the expression $\eta \times \theta$ means the horizontal composition of the natural transformations η and θ rather than the vertical composition.

Remark 5.2.2. Observe the formal similarity of the triangles of equation (2) to the ones of equation (1). This is a good indication that adjoints are 'higher duals'.

Proposition 5.2.3. *Adjoints, if they exist, are unique up to unique isomorphism.*

The category **Cat** is the prototypical example of a 2-category: the objects of **Cat** are all (small) categories, the 1-morphisms are functors, and the 2-morphisms are natural transformations. Definition 5.2.1 didn't really rely on any of the properties of **Cat**, and so we can immediately generalise it to any 2-category:

Definition 5.2.4. Let $\mathcal{C}$ be any 2-category. Let X, Y be objects of $\mathcal{C}$ and let $F : X \to Y$ and $G : Y \to X$ be two 1-morphisms. Say that a 2-morphism $u : \mathrm{id}_X \to G \circ F$ is the **unit of an adjunction between** F **and** G if there exists another 2-morphism $v : F \circ G \to \mathrm{id}_Y$ such that the following two triangles commute:

$$\begin{array}{ccc} F \circ \mathrm{id}_X & & \\ {\scriptstyle \mathrm{id}_F \times u}\downarrow & \searrow^{\mathrm{id}_F} & \\ F \circ G \circ F & \xrightarrow[v \times \mathrm{id}_F]{} & \mathrm{id}_Y \circ F \end{array} \qquad \begin{array}{ccc} \mathrm{id}_X \circ G & & \\ {\scriptstyle u \times \mathrm{id}_G}\downarrow & \searrow^{\mathrm{id}_G} & \\ G \circ F \circ G & \xrightarrow[\mathrm{id}_G \times v]{} & G \circ \mathrm{id}_Y \end{array}$$

In this case we say that v is the **counit**, and that F (resp. G) is **left** (resp. **right**) adjoint to G (resp. F).

[13] Once again, these triangles are really pentagons. If we think of **Cat** as a strict 2-category, then they are squares since we don't need any associators.

Remark 5.2.5. If u is the unit of an adjunction, then a compatible counit v is uniquely determined, and vice versa. So it's enough to specify the existence of either u or v.

Example 5.2.6. A category with a single object is the same thing as a monoid. Similarly if $\mathcal{C}$ is a 2-category with a single object $*$ then the category $\mathrm{Hom}_{\mathcal{C}}(*, *)$ is a monoidal category.

Conversely if M is a monoidal category then we can build a 2-category $\mathcal{B}M$ with a single object *, hom-category $\mathrm{Hom}_{\mathcal{B}M}(*, *) \cong M$ and composition law for 1-morphisms given by the tensor product on M.

Then an object X of M is right dual to an object Y of M if and only if it is right adjoint to Y when both are considered as 1-morphisms of $\mathcal{B}M$. We often call $\mathcal{B}M$ the **delooping** of M.

Adjoints are closely related to invertibility:

Proposition 5.2.7. *Let $\mathcal{C}$ be a 2-category in which every 2-morphism is invertible. Let f be a 1-morphism of $\mathcal{C}$. Then the following are equivalent:*

i) f is invertible.

ii) f admits a left adjoint.

iii) f admits a right adjoint.

Definition 5.2.8. Say that a 2-category $\mathcal{C}$ **has adjoints for 1-morphisms** if every 1-morphism has both a left and a right adjoint.

5.3 Adjoints in higher categories

We'd like to generalise Definition 5.2.4 from 2-categories to higher categories.

Definition 5.3.1. Let $n \geq 2$ and let $\mathcal{C}$ be an (∞, n)-category. Let $h_2\mathcal{C}$ be the **homotopy 2-category** of $\mathcal{C}$, with

$$\begin{aligned} \text{objects} &\longleftrightarrow \text{objects of } \mathcal{C} \\ \text{1-morphisms} &\longleftrightarrow \text{1-morphisms of } \mathcal{C} \\ \text{2-morphisms} &\longleftrightarrow \text{isomorphism classes of 2-morphisms of } \mathcal{C} \end{aligned}$$

Remark 5.3.2. **Homotopy n-categories** are defined similarly.

Definition 5.3.3. Let $\mathcal{C}$ be an (∞, n)-category. Say that $\mathcal{C}$ **has adjoints for 1-morphisms** if $h_2\mathcal{C}$ has adjoints for 1-morphisms. More generally, for $1 < k < n$ say that $\mathcal{C}$ **has adjoints for k-morphisms** if for any two objects X, Y of $\mathcal{C}$ the $(\infty, n-1)$-category $\mathrm{Map}(X, Y)$ has adjoints for $(k-1)$-morphisms. Say that $\mathcal{C}$ **has adjoints** if it has adjoints for k-morphisms for all $0 < k < n$.

Remark 5.3.4. If every k-morphism in $\mathcal{C}$ is invertible then $\mathcal{C}$ has adjoints for k-morphisms. The converse is true provided that all $(k+1)$-morphisms are invertible - this is a consequence of Proposition 5.2.7.

Remark 5.3.5. The condition that $\mathcal{C}$ have adjoints depends on the choice of n. If we regard $\mathcal{C}$ as an $(\infty, n+1)$-category with all $(n+1)$-morphisms invertible then in general $\mathcal{C}$ does not have adjoints for n-morphisms unless it is an ∞-groupoid.

If $\mathcal{C}$ is monoidal then we can ask for a bit more:

Definition 5.3.6. Let $\mathcal{C}$ be a monoidal (∞, n)-category. Say that $\mathcal{C}$ **has duals** if the following two conditions are satisfied:

i) Every object X has both a left and a right dual when considered as an object of the homotopy category $h\mathcal{C}$.[14]

ii) $\mathcal{C}$ has adjoints.

Remark 5.3.7. We can generalise our earlier construction of Example 5.2.6. If $\mathcal{C}$ is a monoidal (∞, n)-category then it is possible to build an $(\infty, n+1)$-category $\mathcal{BC}$ (the **delooping** of $\mathcal{C}$) with a single object $*$, recovering $\mathcal{C}$ as the mapping object $\mathrm{Hom}_{\mathcal{BC}}(*, *)$. Then $\mathcal{C}$ has duals if and only if $\mathcal{BC}$ has adjoints.

5.4 Full dualisability

Given a symmetric monoidal (∞, n)-category we'd like to pick out the largest subcategory with duals.

Theorem 5.4.1. *Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category. Then there exists another symmetric monoidal (∞, n)-category $\mathcal{C}^{\mathrm{fd}}$ with duals, and a symmetric monoidal functor $i : \mathcal{C}^{\mathrm{fd}} \to \mathcal{C}$, universal among symmetric monoidal functors $j : \mathcal{D} \to \mathcal{C}$ where $\mathcal{D}$ has duals.*

Remark 5.4.2. $\mathcal{C}^{\mathrm{fd}}$ is determined up to equivalence by the above properties. In general we can obtain $\mathcal{C}^{\mathrm{fd}}$ from $\mathcal{C}$ by repeatedly discarding morphisms that don't admit adjoints (and objects that don't admit duals).

Example 5.4.3. If $\mathcal{C}$ is a symmetric monoidal $(\infty, 1)$-category then $\mathcal{C}^{\mathrm{fd}}$ is equivalent to the full subcategory of $\mathcal{C}$ spanned by the dualisable objects.

Definition 5.4.4. Say that an object X of $\mathcal{C}$ is **fully dualisable** if it belongs to the essential image[15] of the functor i.

Example 5.4.5. For each $n > 0$, the (∞, n)-category $\mathbf{Bord}_n^{\mathrm{fr}}$ has duals. Every k-morphism can be identified with an oriented manifold M; the morphism $\bar{M}$ (M with the opposite orientation) is both a left and a right adjoint to M.

[14] Note that $h\mathcal{C}$ inherits its monoidal structure from $\mathcal{C}$. If $\mathcal{C}$ is symmetric monoidal then this condition is the condition that every object be dualisable.

[15] Recall that the **essential image** of a functor $F : \mathcal{D} \to \mathcal{E}$ is the smallest isomorphism-closed subcategory of $\mathcal{E}$ containing the image of F.

Example 5.4.6. If $\mathcal{C}$ is the $(\infty, 1)$-category $\mathbf{Vect}_k$, then an object of $\mathcal{C}$ is fully dualisable if and only if it is finite-dimensional.

This generalises to the following:

Proposition 5.4.7. *An object of a symmetric monoidal $(\infty, 1)$-category is fully dualisable if and only if it is dualisable.*

In general full dualisability is a much stronger condition than dualisability! In dimension 2, there are some simple criteria for testing whether or not an object is fully dualisable:

Proposition 5.4.8. *Let $\mathcal{C}$ be a symmetric monoidal $(\infty, 2)$-category. Let X be an object of $\mathcal{C}$. Then X is fully dualisable if and only if it admits a dual $X^\vee$ and the evaluation map $\mathrm{ev} : X \otimes X^\vee \to 1$ has both a left and a right adjoint.*

6 The Cobordism Hypothesis

In this short section we rigourously state the Cobordism Hypothesis. We begin with some bookkeeping.

6.1 Terminology

Definition 6.1.1. An (∞, n)**-functor** between two (∞, n)-categories $\mathcal{C}$ and $\mathcal{D}$ is a map of the underlying simplicial spaces (which is itself a natural transformation between the defining functors).

Theorem 6.1.2. *The collection $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ of (∞, n)-functors between two (∞, n)-categories itself forms an (∞, n)-category.*

Remark 6.1.3. The collection of all (small) (∞, n)-categories naturally forms an $(\infty, n+1)$-category with mapping objects $\mathrm{Map}(\mathcal{C}, \mathcal{D}) = \mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Proposition 6.1.4. *We can also define* ***symmetric monoidal*** (∞, n)***-functors*** *between symmetric monoidal (∞, n)-categories. The collection of symmetric monoidal (∞, n)-functors between two symmetric monoidal (∞, n)-categories $\mathcal{C}$ and $\mathcal{D}$ itself forms an (∞, n)-category, which we refer to as $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$.*

Definition 6.1.5. A **fully extended framed n-dimensional topological field theory** is a symmetric monoidal (∞, n)-functor with source $\mathbf{Bord}_n^{\mathrm{fr}}$. The collection of all fully extended framed n-TFTs with target $\mathcal{C}$ is the (∞, n)-category $\mathrm{Fun}^{\otimes}(\mathbf{Bord}_n^{\mathrm{fr}}, \mathcal{C})$.

Definition 6.1.6. Given an (∞, n)-category $\mathcal{C}$, I will denote the underlying $(\infty, 0)$-category[16] of $\mathcal{C}$ by $\pi_{\leq \infty}(\mathcal{C})$. This notation is not standard.

[16]a.k.a. ∞-groupoid

6.2 A Precise Statement

Claim 6.2.1 (the Cobordism Hypothesis). *If $\mathcal{C}$ is a symmetric monoidal (∞, n)-category then the evaluation functor $Z \mapsto Z(*)$ induces an equivalence*

$$\mathrm{Fun}^{\otimes}(\mathbf{Bord}_n^{\mathrm{fr}}, \mathcal{C}) \xrightarrow{\sim} \pi_{\leq\infty}(\mathcal{C}^{\mathrm{fd}})$$

In particular, the Cobordism Hypothesis states that $\mathrm{Fun}^{\otimes}(\mathbf{Bord}_n^{\mathrm{fr}}, \mathcal{C})$ is an ∞-groupoid, and hence a topological space. We can think of it as a classifying space for fully dualisable objects in $\mathcal{C}$.

It is not too difficult to prove that $\mathrm{Fun}^{\otimes}(\mathbf{Bord}_n^{\mathrm{fr}}, \mathcal{C})$ is an ∞-groupoid. The hard part of proving the Cobordism Hypothesis is proving that the induced functor is an equivalence. A sketch proof of this is given by Lurie in [1].

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