Introduction to Factorization Homology

What is F.H.?

Diff Geom

$H_i(\mathbb{R}^n) \cong H_i(\{*\})$
is there a homology thy better suited for C^∞-mfds?

Higher Algebra

A-alg, $M \in A$-mod $\rightsquigarrow HH_*(A, M)$
what is $HH_*(\mathcal{C}^\otimes, ?)$?

Factorization Homology

Cobordism Hypothesis
(how to compute n-TFT from some suitably dualizable object)

§1 E_n-algebras or Coefficients

Def^n Disk_1^{or} the topological category

$$\mathrm{ob} = \{ \emptyset, \mathbb{R}, \mathbb{R} \sqcup \mathbb{R}, \dots \} \qquad \mathbb{R}\text{-oriented}$$

$$\mathrm{Map}\left(\coprod_i \mathbb{R}, \coprod_j \mathbb{R}\right) = \{ \text{open } C^\infty\text{-embeddings } \mathbb{R}^{\sqcup i} \hookrightarrow \mathbb{R}^{\sqcup j} \text{ that resp. orientation} \}$$

Rmk
- $\sqcup$ is a symm. mon. str.
- Hom spaces inherit topology from compact-open
- paths are isotopies

Def^n An E_1- (or Disk_1^{or}-) algebra in $\mathrm{Vect}_k^{\otimes}$ is a symm. mon. functor

$$A : (\mathrm{Disk}_1^{or})^{\sqcup} \longrightarrow \mathrm{Vect}_k^{\otimes}$$

respecting the topology of hom-spaces.

What is that?

$\mathrm{Hom}_{\mathrm{Vect}_k}(V,W)$ has discrete topology

thus $f \overset{\text{isot.}}{\simeq} g \Rightarrow A(f) = A(g)$

$A(\emptyset) = k$

$A(\mathbb{R}) = V$

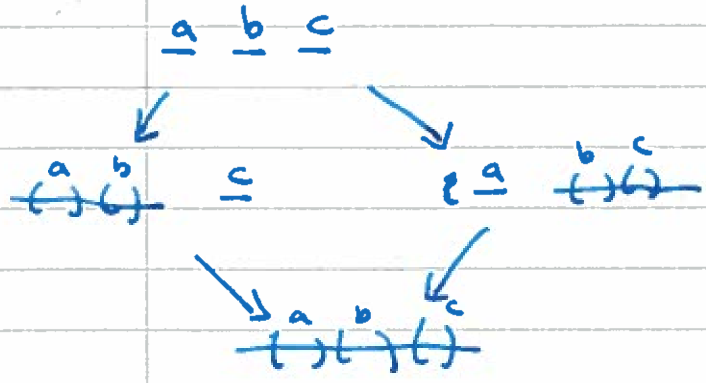
$A(\mathbb{R}^{\sqcup i}) = V^{\otimes i}$

Fact: any embed. $\mathbb{R} \hookrightarrow \mathbb{R} \overset{\text{isot.}}{\simeq} \mathbb{R} \xrightarrow{id} \mathbb{R}$

$A(\mathbb{R} \to \mathbb{R}) = id_V$

$$\begin{tikzcd} \underset{a}{\mathbb{R}} \sqcup \underset{b}{\mathbb{R}} \arrow[r] \arrow[d, "swap"'] & \mathbb{R} \\ \underset{b}{\mathbb{R}} \sqcup \underset{a}{\mathbb{R}} \arrow[ur] & \end{tikzcd} \quad \overset{A}{\longmapsto} \quad \begin{tikzcd} V \otimes V \arrow[r, "m"] \arrow[d, "swap"'] & V \\ V \otimes V \arrow[ur, "m'"'] & \end{tikzcd}$$

$$\begin{tikzcd} \emptyset \sqcup \mathbb{R} \arrow[r] \arrow[d] & \mathbb{R} \\ \mathbb{R} \sqcup \mathbb{R} \arrow[ur] & \end{tikzcd} \quad \overset{A}{\longmapsto} \quad \begin{tikzcd} k \otimes V \arrow[r] \arrow[d] & V \\ V \otimes V \arrow[ur, "m"'] & \end{tikzcd}$$



$\overset{A}{\longmapsto}$

$$\begin{tikzcd} & V \otimes V \otimes V \arrow[dl, "m \otimes id"'] \arrow[dr, "id \otimes m"] & \\ V \otimes V \arrow[dr, "m"'] & & V \otimes V \arrow[dl, "m"] \\ & V & \end{tikzcd}$$

Prop $E_1\text{-alg}(\mathrm{Vect}^{\otimes}) \simeq \mathrm{Assoc}(\mathrm{Vect}^{\otimes})$

Generalize: $\mathrm{Vect}_k^{\otimes} \longrightarrow \mathcal{C}^{\otimes}$

$\mathrm{Disk}_1^{or} \longrightarrow \mathrm{Disk}_n^{or}$

Rmk or $\longrightarrow$ more interesting tangential str.

Examples
- $E_1\text{-alg}(\mathrm{Cat})$ = tensor cats

  $E_2\text{-alg}(\mathcal{C}^{\otimes})$ is $A : \mathrm{Disk}_2^{or} \longrightarrow \mathcal{C}^{\otimes}$ $\quad$ $\emptyset, \mathbb{R}^2, \mathbb{R}^2 \sqcup \mathbb{R}^2$, etc
- $\mathcal{C} = \mathrm{Vect}^{\otimes}$ diagrams

$$(a)\ (b) \longrightarrow ((a)\ (b)) \overset{A}{\longmapsto} V \otimes V \longrightarrow V$$

2 | isot.

$$((b)\ (a))$$ thus comm. algebras.

Def^n A locally constant factorization algebra on $\mathbb{R}^n$ in $\mathcal{C}^{\otimes}$ is a

precosheaf $\mathcal{F} : \mathrm{Op}(\mathbb{R}^n) \longrightarrow \mathcal{C}$ s.t.

(factorization) disjoint U_i: $\mathcal{F}(\bigsqcup_i U_i) \simeq \bigotimes_i \mathcal{F}(U_i)$

(locally constant) $U \subset V$ retract then $\mathcal{F}(U) \xrightarrow[\sim]{q.i.} \mathcal{F}(V)$

Rmk
- Can define l.c.f.a for any manifold, but need an extra gluing (cosheaf) condition
- L.C.f.a. should model observables in a QFT.

Ex $A \in E_1\text{-alg}(\mathcal{C}^{\otimes})$ defines l.c.f.a. $U \longmapsto A(U)$ on $\mathbb{R}$

Thm (Lurie) $\quad E_n\text{-alg}(\mathcal{C}^{\otimes}) \simeq \mathrm{Fact}^{l.c.}_{\mathbb{R}^n}(\mathcal{C}^{\otimes})$ $\square$

Ex $\mathcal{C}^{\otimes} = \text{dg-Vect}^{\overset{L}{\otimes}}$ $\qquad E_1$-alg $\qquad A$

$V = A(\mathbb{R}) \qquad \underset{}{\overset{a}{-}}\ \underset{}{\overset{b}{-}} \hookrightarrow \overset{a}{(\ )}\overset{b}{(\ )} \qquad V \overset{L}{\otimes} V \to V$

and associativity is only satisfied up to quasi-isomorphism!

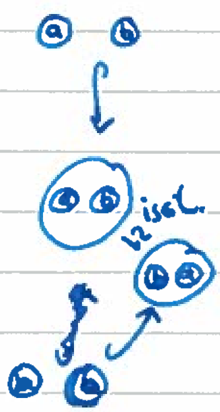
$E_1(\text{dg-Vect}^{\overset{L}{\otimes}}) = A_\infty$-algebra (homotopy algebra)

Thm $E_1 \overset{\sim}{=} Ass_\infty$ (as ∞-operads). □

~~Ex~~ What is an monoidal ∞-cat? E_1-alg(∞-Cat).

Ex $Cat^{\boxtimes}$ (2-cat of cats; 0-cats, 1-functors, 2-nat tr.)

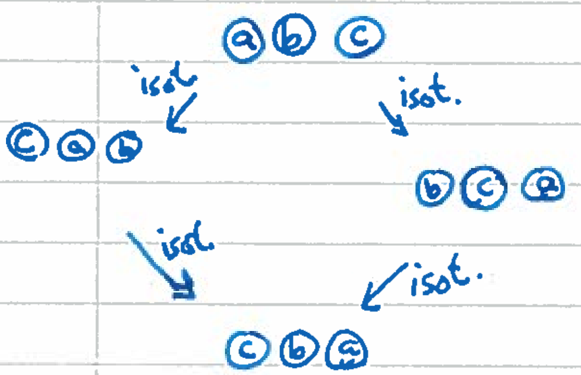
E_2-alg: $A(\mathbb{R}^2) = \mathcal{C} \in Cat$.



$\overset{A}{\longmapsto} \quad \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}$ $\qquad$ (thus $\mathcal{C} = \mathcal{C}^{\otimes}$)

$\Downarrow$ nat.tr. R

$\overset{A}{\longmapsto} \quad \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}$



$\overset{A}{\longmapsto}$ R satisfies the conditions of being a braiding on $\mathcal{C}$ (up to a 3-morphism actually, but those are all identities)

etc

E_2-alg(Cat) = braided mon. cat

§2 Left Kan Extensions or Globalization

$$\begin{tikzcd}
\mathrm{Disk}_n^{or} \arrow[r, "A"] \arrow[d, hook, "I"'] & \mathcal{C}^{\otimes} \\
\mathrm{Mfd}_n^{or} \arrow[ur, dotted, "?"'] &
\end{tikzcd}$$

Recall: colimit (direct limit) is a universal object known through maps out of it.

__Ex__ (Top) $X \cup_f Y$

$$\begin{tikzcd}
X \cup_f Y \arrow[dr, dotted, "\exists!"] & & Y \arrow[ll] \arrow[dl] \\
& T & \\
X \arrow[uu] \arrow[ur] & & A \arrow[ll, hook'] \arrow[uu, "f"'] \arrow[ul]
\end{tikzcd}$$

$$\begin{matrix} X \\ \cup \\ A \end{matrix} \xrightarrow{f} Y$$

Colimit ~ quotients of disjoint unions (gluing)
Manifolds = glued disks

$$\varinjlim \left(I \downarrow M \xrightarrow{P} \mathrm{Disk}_n^{or} \xhookrightarrow{I} \mathrm{Mfd}_n^{or} \right) = M$$

where $I \downarrow M$ obj $I(D) \to M$

mor

$$\begin{tikzcd}
I(D) \arrow[r] \arrow[d, "I(f)"'] & M \\
I(D') \arrow[ur] &
\end{tikzcd} \qquad f : D \to D'$$

and $P(I(D) \to M) = D$.

In English: $I \downarrow M$ contains all info of how disks map into M.
If you know $f_i|_{D_i \subset M} \to M'$ & f_i glue $\rightsquigarrow f : M \to M'$

The colimit expresses that M can be seen as glued disks in a __canonical way__ (say the category of disks is __dense__ in manifolds).

Def^n Given $A : \mathrm{Disk}_n^{or} \to \mathcal{C}^{\otimes}$ define the LKE of A along I

$$L_I A(M) := \varinjlim \left(I \downarrow M \xrightarrow{P} \mathrm{Disk}_n^{or} \xrightarrow{A} \mathcal{C} \right).$$

Rmk
- Of course LKE need not exist in general
- One can also define $L_I A$ through a universal property as extending

$$\begin{tikzcd} \mathcal{D} \arrow[r, "A"] \arrow[d, hook, "I"'] & \mathcal{C} \\ \mathcal{M} \arrow[ur, dotted, "L_I A"'] \arrow[u, phantom, shift left=6, "\Downarrow"] \end{tikzcd}$$

but this def^n is better

Ex $L_I I(M) = M$. ($L_I I = \mathrm{id}_M$ exactly expresses $I : \mathcal{D} \hookrightarrow \mathcal{M}$ dense)

Notation / Def^n

$$\begin{tikzcd} \mathrm{Disk}_n^{or} \arrow[r, "A"] \arrow[d, hook] & \mathcal{C}^{\otimes} \\ \mathrm{Mfd}_n^{or} \arrow[ur, "\int_{\bullet} A"'] \end{tikzcd}$$

Factorization Homology

so $L_I A(M) =: \int_M A$

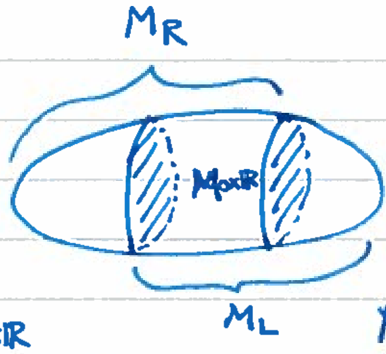
§3 Examples of Factorization Homology

Ex
- $\int_{\mathbb{R}^n} A = A$ (this follows from formal property of LKE along nice functors.
  I is fullyfaithful $\Rightarrow L_I A \circ I \cong A$

Fact: $U \mapsto \int_U A$ defines a l.c.f.a. on M

Problem: As a colimit, difficult to compute (only know it through maps out of it).

Let $M = M_R \cup_{M_0 \times \mathbb{R}} M_L$



have embedding

makes $\int_{M_R} A$ a right $\int_{M_0 \times \mathbb{R}} A$ module

similarly $\int_{M_L} A$ a left —"—

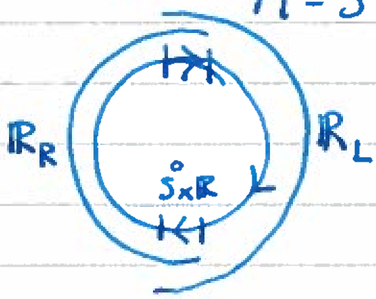
Thm (Lurie, Francis)

$$\int_M A \simeq \int_{M_R} A \bigotimes_{\int_{M_0\times\mathbb{R}} A} \int_{M_L} A \qquad \text{(Excision)}$$

$\square$

Example $A \in A_\infty\text{-alg}$ $\left(= E_1\text{-alg}\left(dg\text{-}Vect^{\otimes^L}\right)\right)$

$M = S^1$



$$\int_{S^1} A \simeq \int_{\mathbb{R}_R} A \bigotimes_{\int_{S^0\times\mathbb{R}} A} \int_{\mathbb{R}_L} A$$

$$\simeq A \overset{L}{\otimes}_{A \otimes A^{op}} A$$

$$=: CH_*(A) \qquad \text{(Hochschild Chains)}$$

Cor $S^1 \circlearrowright S^1$ induces action $S^1 \circlearrowright \int_{S^1} A \simeq CH_*(A)$ $\square$

Def^n Let a manifold homology theory be a symm. mon. functor valued in $\mathcal{C}^{\otimes}$

$$\mathcal{M}fd_n \longrightarrow \mathcal{C}^{\otimes n}$$

satisfying excision

Thm (Ayala-Francis)

There is an equivalence

$$\int : E_n\text{-alg}(\mathcal{C}^{\otimes}) \rightleftarrows H(\mathcal{M}fd, \mathcal{C}^{\otimes}) : ev_{\mathbb{R}^n} \qquad \square$$

Rmk • Compare to Eilenberg - Steenrod

$$H^{sing} : Ab \rightleftarrows H(Top, GrAb)$$

• For M^m, N^n $(m \neq n)$

$$\int_{M^m} A \overset{?}{=} \int_{N^n} A \quad \text{does not even make } \underline{\text{sense}}$$

so f.h. sees different dimension ($\mathbb{R}^n$ vs $\mathbb{R}^m$)

• Let $\mathcal{U}_q\mathfrak{g}$ denote $Rep(\mathcal{U}_q\mathfrak{g})$, it is braided thus E_2.

One can now ask questions such as

$$\int_{\Sigma_g} \mathcal{U}_q\mathfrak{g} = ?$$

(Σ_g - genus g orientable surface)