Lecture 1: Origins of Topological Field Theory

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Quantum Field Theory

We start with some manifold which may be a 4-manifold M^4 equipped with a metric with signature $(+\ +\ +-)$ ('space time'), or a 3-manifold M^3 equipped with a Riemannian metric, or a surface $\Sigma = M^2$ with a conformal structure. On this manifold M we study some space of 'fields' $\mathcal{F}$. That is, we have some natural bundles

$$\begin{array}{c} \xi \\ \downarrow \\ M \end{array}$$

and we wish to study the sections $\mathcal{F} = \Gamma(M, \Omega^*\xi)$. In physical situations there is a Lagrangian

$$\mathcal{L} = f\left(\varphi, \frac{\partial \varphi}{\partial x_i}, \frac{\partial^2 \varphi}{\partial x_i \partial x_j}, \dots\right),\ f \in \mathbb{C}[x_1, x_1, \dots],$$

and the main object of study is the 'partition function'

$$Z = \int_{\varphi \in \Gamma(M, \Omega^*\xi)} e^{\pi i \int_M \mathcal{L}(\varphi)}\, d\varphi.$$

This is the **measure** against which one compares correlations of fields ψ_1, ψ_2 i.e. the likelihood of observing ψ_1 given ψ_2 and vice versa. These integrals are ill-posed in general, not only divergent, but underfined because $d\varphi$ does not exist as stated. QFT addresses this in various ways to still get meaningful physical quantities out. The partition function Z depends, a priori, on the extra data on M, such as metric. However, in many field theories, especially 'super symmetric' ones, this dependence is trivial, and the field theory is said to be **topological**.

Example 1. *An example of a topological quantum field theory is based on the Cern-Simons functional on a 3-manifold M^3 with Riemannian metric. We have* *a trivial $\mathfrak{G}$-bundle over M where $\mathfrak{G}$ is a semi-simple lie algebra. So we have*

$$\mathcal{F} = \Gamma(M, \Omega'(\mathfrak{G})).$$

Let the connection be $\nabla = A_i(m)dx_i$, then in local coordinates

$$\mathcal{L}(\nabla) = tr(F \wedge A - \frac{1}{3} A \wedge A \wedge A)$$

where $F = dA + A \wedge A$ is the curvature of the connection.

$$\Omega^3(M, \mathfrak{G} \otimes 3) \xrightarrow{tr} \mathcal{L}(\nabla) \in \Omega^3(M^3).$$

Topological Quantum Field Theories

In the 1980s Atiyah and Segal realised that the formal properties of the (ill-defined) partition function may nerveless lead to interesting **invariants** leading to an axiomatisation of TQFTs.

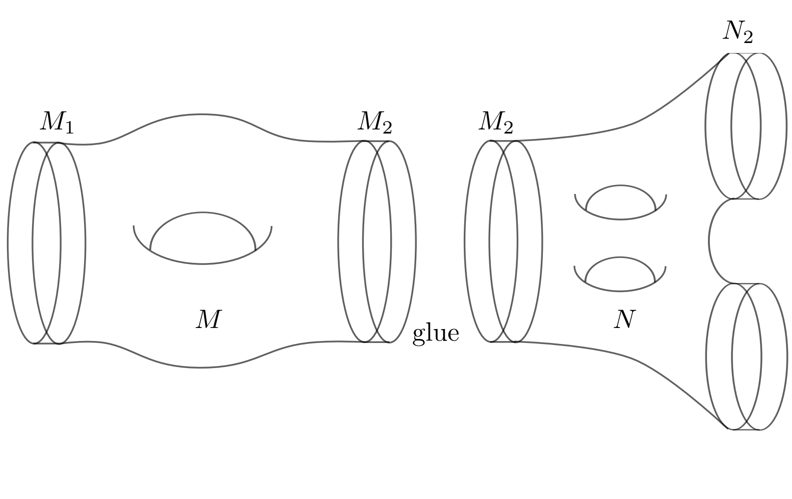
In particular, they were interested in manifold invariants which exploited the **manifold** nature, not just the topological space. The most basic properties are those of **boundary conditions** and compatibility with **cobordisms** and **disjoint unions**.

1. Given submanifolds $M_1, M_2 \subseteq \partial M$ and fields supported in some collar neighbourhoods $M_1 \times I \subseteq M$, $M_2 \times I \subseteq M$ we can define

$$Z(\varphi_1, \varphi_2) = \int_{\substack{\varphi \in \Gamma(M.\Omega^* \xi) \\ \varphi|_{M_1 \times I} = \varphi \\ \varphi|_{M_2 \times I} = \varphi_2}} e^{i\pi \int_M \mathcal{L}(\varphi)} \, d\varphi$$

   This 'defines' a map

$$Z : \mathcal{F}(M_1 \times I) \to \mathcal{F}(M_2 \times I).$$



2. These compose under gluing of oriented **cobordisms**.

$$Z(N) \circ Z(M) = Z(M \text{ glued to } N)$$

A cobordism from M_1 to M_2 (both $(n-1)$-manifolds) is a n-manifold M with boundary $\partial M = M_1 \sqcup M_2 = M_{in} \sqcup M_{out}$. A manifold is orientable if $\Lambda^*(TM)$ is trivial and an orientation is decomposition of this line into $\pm$.

3. Let M, N be manifolds and $M_1, M_2 \subseteq \partial M$.

$$Z(M \sqcup N) = Z(M) \cdot Z(M)$$

$$Z(M_1 \sqcup M_2) = Z(M_1) \otimes Z(M_2)$$

Atiyah and Segal studied the **cobordism category** $Cob^1(n)$. The cobordism category has

Objects: $(n-1)$-dimensional, compact, **oriented**, closed manifolds.

Morphisms: $Hom(M_1, M_2)$ consists of diffeomorphism classes of oriented cobordisms M^n with $\partial M = M_{in} \sqcup M_{out} = M_1 \sqcup M_2$. Quotienting by diffeomorphism is required for associativity. Orientation is needed to determine the 'source' and 'target' of cobordisms.

Atiyah and Segal defined an n-dimensional TQFTs as a symmetric monoidal functor

$$Z : Cob^1(n)^{\sqcup} \to Vect_k^{\otimes}.$$

The mondoidal structure on $Cob^1(n)^{\sqcup}$ is given by disjoint union $\sqcup$ and on $Vect_k^{\otimes}$ given by the usual tensor product on vector spaces.

This definition is already very interesting but we will extend it in various ways. The first way is to replace $Vect_k^{\otimes}$ with a general symmetric monoidal category $\mathcal{C}^{\otimes}$. This makes a n-dimensional TQFTs a symmetric monoidal functor

$$Z : Cob^1(n)^{\sqcup} \to \mathcal{C}^{\otimes}.$$

We can also extend the definition to higher categories. Given $k = 1, 2, \ldots, \infty$, will see a natural category, and study functors

$$Z : Cob^k(n)^{\sqcup} \to \mathcal{C}^{\otimes},$$

where $\mathcal{C}^{\otimes}$ is some natural higher category. For example

$$\mathcal{C}^{\otimes} = \mathrm{Alg}_k^2 = \begin{cases} k \text{ algebras as objects} \\ A - B \text{ bimodules as morphisms} \\ \text{bimodule maps as 2-morphisms} \end{cases}$$

or

$$\mathcal{C}^{\otimes} = \mathrm{Alg}^2_{\mathrm{Vect}_k} = \begin{cases} \text{tensor categories as objects} \\ \text{bimodule categories as 1-morphisms} \\ \text{natural transformations as 3-morphisms} \end{cases},$$

or $\mathcal{C}^{\otimes}$ is a braded tensor category or $\mathcal{C}^{\otimes}$ is a modular tensor category.

The Cobordism Hypothesis

We shall also see the **cobordism hypothesis**: Roughly this says that we can find 'generators' and 'relations' presentations for $Cob^k(n)$ and hence classify TQFTs in terms of special objects in $\mathcal{C}^{\otimes}$ which are called (fully) dualizable. The hypothesis hypothesis has been worked on my many people including Baez-Dolan (conjecture), Lurie (proof), Scheimbauer (segal spaces) and Ayala-Francis Tannaka (factorisation homology).

Cobordism hypothesis:

- $Cob^{\infty}(n)$ is generated by a single object, the n-disc. Hence a fully extended TFT is completely determined by $Z(\mathbb{R}^n) \in \mathcal{C}^{\otimes}$.
- $Z(\mathbb{R})$ must be fully dualizable, which asserts the existance of many adjoints and duals in $\mathcal{C}^{\otimes}$ and is essentialy a strong finiteness condition.