Lecture 2: TQFTs in 1 and 2 Dimensions

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Definition 1. *A $(n, n-1)$-dimensional TQFT is a symmetric monoidal functor*

$$Z : Cob^1(n)^{\sqcup} \to \mathcal{C}^{\otimes}$$

we shall assume for simplicity that $\mathcal{C}^{\otimes} = Vect_k^{\otimes}$, the category of vector spaces over a field k equipped with the standard tensor product.

The Category $\mathbf{Cob}^1(n)$

The superscript 1 of $Cob^1(n)$ tells us we are are dealing with a standard category; higher values are used to denote higher categories which will be considered later in this talk and in later talks. As result to describe $Cob^1(n)^{\sqcup}$ we need to know its objects and morphisms.

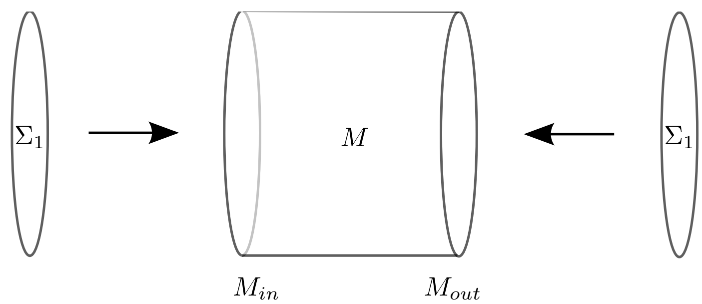
- The objects of $Cob^1(n)$ are $(n-1)$-dimensional manifolds Σ which are closed, compact and oriented.
- The morphisms of $Cob^1(n)$ are oriented cobordisms $\Sigma_0 \overset{M}{\Longrightarrow} \Sigma_1$ up to equivalence of cobordisms.

Definition 2. *An oriented cobordism from Σ_0 to Σ_1 is an oriented n-manifold M together with maps*

$$\Sigma_0 \to M \leftarrow \Sigma_1$$

such that Σ_0 maps diffeomorphically onto M_{in} [1] and Σ_1 maps diffeomorphically onto M_{out}. We denote such a cobordism as $\Sigma_0 \overset{M}{\Longrightarrow} \Sigma_1$.

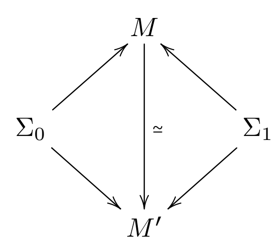
[1] The in and out boundaries of M are determined by how the choice of orientation M corresponds to the choice of orientation on parts of its boundary. We shall not go into detail as to how this is exactly done however the point is to add a directionality to our cobordism morphisms.



Two compatible cobordisms $\Sigma_0 \overset{M}{\Longrightarrow} \Sigma_1$ and $\Sigma_1 \overset{M'}{\Longrightarrow} \Sigma_2$ can be composed by gluing together the n-manifolds along their common boundary Σ_1 [2] to give a new cobordism $\Sigma_0 \overset{M' \circ M}{\Longrightarrow} \Sigma_2$.

This composition operation is not strictly associative. However it is associate up to diffeomorphism relative to the boundaries Σ_0 and Σ_1 which leads to a good definition of equivalence of cobordisms.

Definition 3. *Two cobordisms $\Sigma_0 \overset{M}{\Longrightarrow} \Sigma_1$ and $\Sigma_0 \overset{M}{\Longrightarrow} \Sigma_1$ are equivalent cobordisms if they have the same boundaries Σ_0, Σ_1 and the n-manifold M is diffeomorphic to M'. This can be expressed by saying the following diagram commutes:*



Finally, $Cob^1(n)$ is a monoidal category with operation $\sqcup$, the disjoint union.

(1,0)-TQFTs

A $(1,0)$-dimensional TQFT is a symmetric monoidal functor

$$Z : Cob^1(1)^{\sqcup} \to \mathrm{Vect}_k^{\otimes}.$$

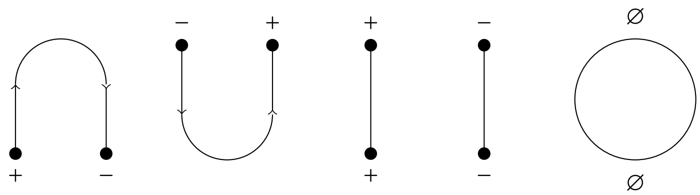
We shall now aim to classify such TQFTs. Unlike in higher dimensions where the aim would be to use TQFTs to understand the category $\mathrm{Cob}^1(n)$, we understand $\mathrm{Cob}^1(1)$ as this just requires knowledge of 0 and 1 dimensional manifolds and

[2] Technically it is only a common boundary up to diffeomorphism so a small collar is needed between M and M'.

we can use this to understand the TQFT. The objects of $\mathrm{Cob}^1(1)$ are disjoint unions of oriented points

$$\left\{ \overset{+}{\bullet}, \overset{-}{\bullet} \right\}.$$

The morphisms of $\mathrm{Cob}^1(1)$ are disjoint unions of



As $Z : Cob^1(1)^{\sqcup} \to \mathrm{Vect}_k^{\otimes}$ is monoidal, we have for disjoint unions $Z(A \sqcup B) = Z(A) \otimes Z(B)$ where either both A, B are 0-manifolds or they are both cobordisms of 0-manifolds. As it a result it is sufficient to define Z on the two points and five cobordisms given above.

$$Z\left(\overset{+}{\bullet}\right) = V_+ \in \mathrm{Vect}_k^{\otimes}$$

$$Z\left(\overset{-}{\bullet}\right) = V_- \in \mathrm{Vect}_k^{\otimes}$$

$$Z\left(\underset{+\sqcup-}{\bigcap}\right) : V_+ \otimes V_- \to k$$

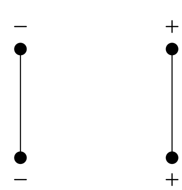
$$Z\left(\overset{-\sqcup+}{\bigcup}\right) : k \to V_- \otimes V_+$$

$$Z\left(\overset{-}{\underset{-}{|}}\right) : V_- \to V_-$$

$$Z\left(\overset{+}{\underset{+}{|}}\right) : V_+ \to V_+$$

$$Z\left(\bigcirc\right) : k \to k$$

The first thing to notice is that



are the identity cobordisms hence must be sent to identity maps in $\mathrm{Vect}_k^{\otimes}$ by the functor Z. Hence

$$Z\left(\begin{smallmatrix}-\\|\\-\end{smallmatrix}\right) = Id_{V_-}$$

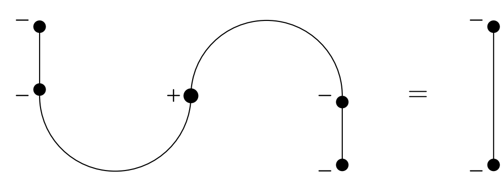
$$Z\left(\begin{smallmatrix}+\\|\\+\end{smallmatrix}\right) = Id_{V_+}$$

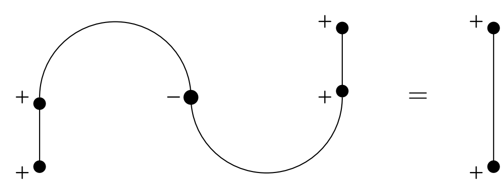
The map $Z\left(\underset{+\sqcup-}{\bigcap}\right) : V_+ \otimes V_- \to k$ gives a map $V_- \to Hom(V_+, k)$ and if this map is bijective then V_- must be the vector dual of V_+ and thus it would be sufficient to specify $\left(\overset{+}{\bullet}\right) = V_+$ only. This is in fact the case and follows from Zorro's Lemma.

Proposition 1 (Zorro's Lemma [3])**.** *For any pair of basis vectors $v_i \in V_-$ and $e_j \in V_+$,*

$$Z\left(\underset{+\sqcup-}{\bigcap}\right)(v_i, e_j) = \delta_{ij}.$$

Proof.





The map $Z\left(\overset{-\sqcup+}{\bigcup}\right) : k \to V_- \otimes V_+$ must have the form $k \mapsto \sum_{i=1}^{n} v_i \otimes e_j$ for some finite number of basis vectors $v_i \in V_-$ and $e_j \in V_+$. Consider a vector $e_j \in V_+$ under the action of the maps of the LHS of the lower diagram:

$$e_j = e_j \otimes k \mapsto e_j \otimes \sum_{i=1}^{n} v_i \otimes e_i \mapsto \sum_{i=1}^{n} e_i \left(Z\left(\underset{+\sqcup-}{\bigcap}\right)(v_i, e_j)\right).$$

The result must be the same as when we act on e_j by the map on the RHS which is just the identity map:

$$e_j \mapsto e_j.$$

[3]Zorro's Lemma also shows that $Z(\bigcup)$ and $Z(\bigcap)$ are adjoints.

Hence we have that

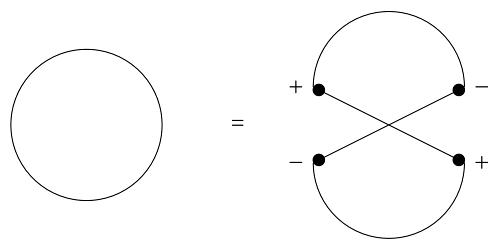
$$\sum_{i=1}^{n} e_i \underbrace{\left(Z\left(\underset{+\sqcup -}{\bigcap} \right)(v_i, e_j) \right)}_{\in k} = e_j$$

which implies that

$$Z\left(\underset{+\sqcup -}{\bigcap} \right)(v_i, e_j) = \delta_{ij}$$

as required. □

We have shown that V_+ is a finite dimensional vector space and $V_- = (V_+)^*$. Finally we see that:



Hence

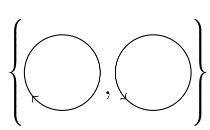
$$(\bigcirc) : 1 \mapsto \sum_{i=1}^{n} e^i \otimes e_i \mapsto \sum_{i=1}^{n} e^i(e_j) = \dim V_+.$$

(2, 1)- TQFTs

We shall now consider a (2,1)-TQFT which is a symmetric monoidal functor:

$$Z : Cob^1(2) \to \mathrm{Vect}_k^{\otimes}.$$

Again we know explicitly what $Cob^1(2)$ as we have a classification of surfaces. The objects of $Cob^1(2)$ are 1-dimensional closed compact oriented manifolds thus must be disjoint unions of



For the cobordisms we have the basic cobordisms given in Figure 1.

Using the classification of surfaces we can decompose any cobordism into a composition of these basis cobordism, such a decomoposition is some called the 'standard form'. We shall illustrate how this is done with an example. Suppose we have a cobordism $S^1 \sqcup S^1 \sqcup S^1 \xRightarrow{M} S^1 \sqcup S^1 \sqcup S^1 \sqcup S^1$ such that M has genus 2 then the standard form is given in Figure 2.

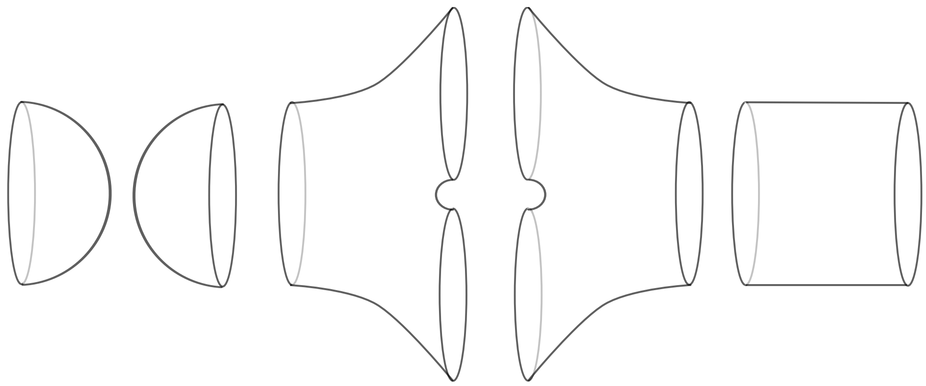


Figure 1: Basic Cobordisms

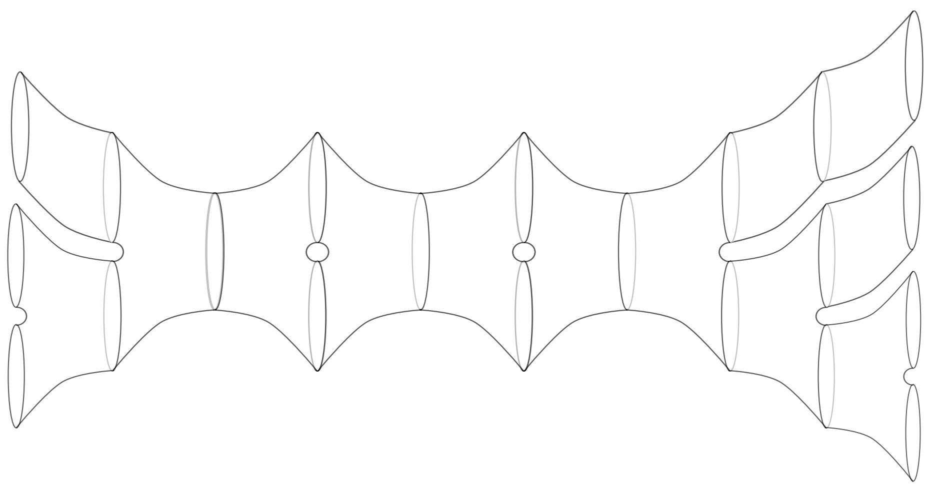


Figure 2: Example of a cobordism in 'standard form'

The first two of the basic cobordisms are used for constructing cobordism which are closed or have a single boundary component.

Firstly, we do not have to specify both orientations of the circle in our TQFT. If we denote $Z(\circlearrowleft) = A \in \mathrm{Vect}_k^{\otimes}$ then $Z(\circlearrowleft) = A^*$, the vector dual of A and A is finite dimensional. This is analgous to the $(1,0)$-TQFT case and the proof uses a higher dimensional version of Zorro's lemma.

Definition 4. *A (unital, associative) k-algebra is a k-vector space A together with two k-linear maps*

$$\mu : A \otimes A \to A \text{ (multiplication)}, \; \eta : k \to A \text{ (unit map)},$$

satisfying the associativity and the unit axiom:

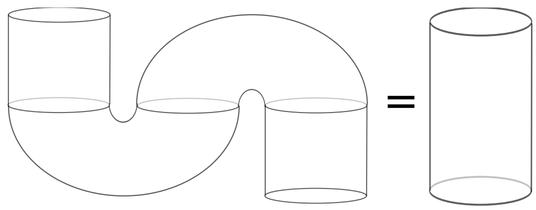


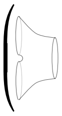
Figure 3: The diagram for Zorro's Lemma in 2-dimensions

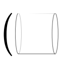
$$(\mu \otimes Id_A)\mu = (Id_A \otimes \mu)\mu$$
$$(\eta \otimes Id_A)\mu = Id_A = (Id_A \otimes \eta)\mu.$$

In other words, a k-algebra is precisely a monoid in the monoidal category $Vect_k^{\otimes}$.

$A = Z(\circlearrowleft)$ is a k-algbera with maps defined:

$\eta = Z\Big($  $\Big) : k \to A$ (unit)

$\mu = Z\Bigg($  $\Bigg) : A \otimes A \to A$ (multiplication)

$Id_A = Z\Big($  $\Big) : A \to A$ (identity)

These maps satisfy associativity and the unit axiom:

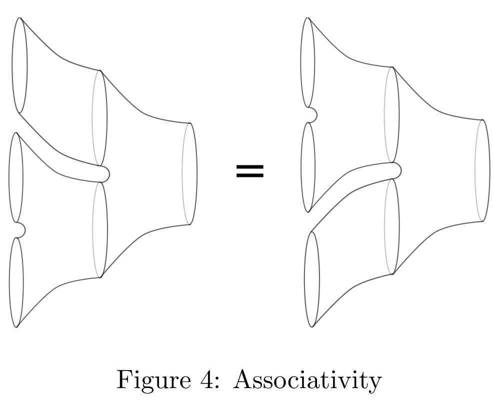


Figure 4: Associativity

Furthermore, A is a commutative algebra:

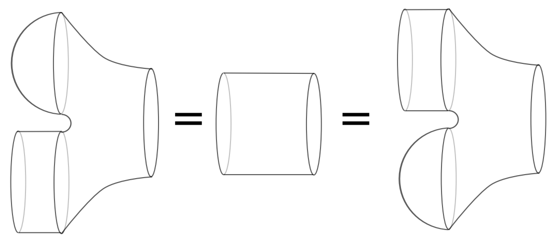


Figure 5: Unit axiom

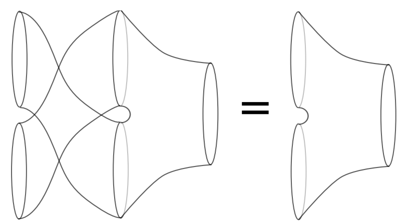


Figure 6: Commutativity

To conclude our classification of $(1,0)$, we shall show that A is a Frobenius algebra.

Definition 5. *A Frobenius algebra is a k-algbera A equipped with an associate non-degenerate pairing $\beta : A \otimes A \to k$ called the Frobenius form.*

The Frobenius form on A is defined as:

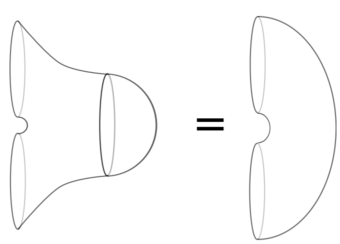


Figure 7: Frobenuis Form

The conditions of associativity and non-degeneracy are encoded in the following two figures.

Another example of a Frobenius algebra is a matrix algebra defined over a field k with Frobenius form $\sigma(a,b) = tr(a \cdot b$. So to summarise:

Proposition 2. *(2,1)-TQFTs are completely determined by their value on $Z\begin{pmatrix} + \\ \bullet \end{pmatrix} = A \in Vect_k^{\otimes}$ where A is a finite dimensional, commutative Frobenius*

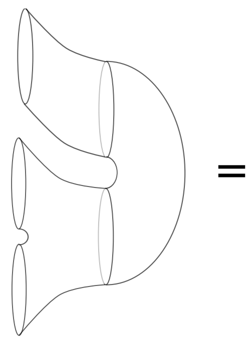


Figure 8: Associativity Condition

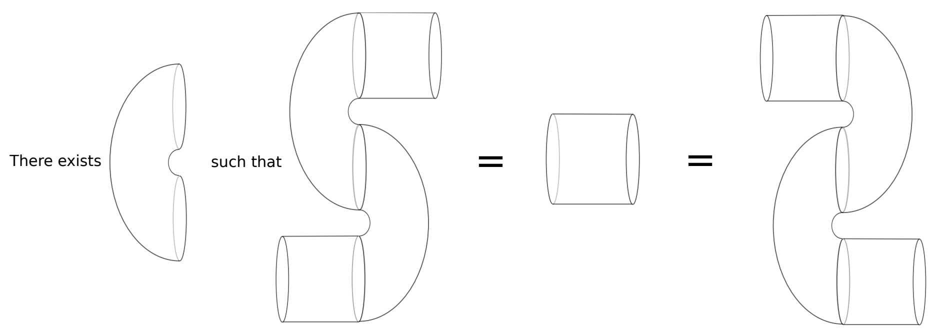


Figure 9: Non-degeneracy Condition

algebra. [4]

Fully extended (2, 1, 0) TQFTs

The definition of a TQFT can be extended to be a functor from higher cobordism categories $Cob^k(n)$ for $k > 1$. How this is done generally will be covered in later talks but in this talk we shall consider an example where $k = 2 = n$.

Definition 6. *A* $(2,1,0)$*-TQFT is a symmetric monoidal functor between weak 2-categories:*

$$Z : Cob^2(2) \to Alg_k^2.$$

We shall begin by defining the 2-category Alg_k^2.

- The objects of Alg_k^2 are k-algebras (k is a field).
- The 1-morphisms of Alg_k^2 are Bimodules.

Definition 7. *An* $A - B$ *bimodule* ${}_AM_B$ *is an abelian groups such that*

[4] This is reversable; every finite dimensional, commutative Frobenius algebra is a TQFT

- *M is a left A module and M is a right B module,*
- *$(am)b = a(mb)$ for all $(a, m, b) \in A \times {}_AM_B \times B$.* [5]

The composition of 1-morphisms is defined for compatible bimodules ${}_AM_B$ and ${}_BN_C$ to be

$$ {}_AM_B \otimes_B {}_BN_C = \frac{{}_AM_B \otimes {}_BN_C}{\langle mb \otimes n - m \otimes bn | m \in {}_AM_B, b \in B, n \in {}_BN_C \rangle} $$

where the unadorned $\otimes$ is the usual tensor product on vector spaces. This composition is only associative up to isomorphisms of bimodules.

- The 2-morphisms of Alg_k^2 are bimodule homomorphisms.

We now need to define the 2-catgory $Cob^2(2)$. We shall actually be considering the **framed**[6] $Cob^2(2)$ category this ensures that gluing works properly.

- The objects of $Cob^2(2)$ are 2-framed points $\left\{ \overset{+}{\bullet}, \overset{-}{\bullet} \right\}$.
- The 1-morphisms of $Cob^2(2)$ are compact 2-framed 1-manifolds.
- The 2-morphisms of $Cob^2(2)$ are compact 2-framed 2-manifolds with corners. [7]

Proposition 3. *Fully extended $(2, 1, 0)$-TQFTs*

$$ Z : Cob^2(2) \to Alg_k^2 $$

are completely determined by their value $B = Z\left(\overset{+}{\bullet} \right) \in Alg_k^2$; which must be a finite dimensional, semi-simple, Frobenuis algebra and $Z(\circlearrowleft)^*$ is the centre of this Frobenius algebra.

We will not have time to give a proof of this result however I shall give a sketch of how the proof works.

If $Z(\overset{+}{\bullet}) = A \in \mathrm{Alg}_k^2$ then $Z(\overset{-}{\bullet}) = A^{op}$. As gluing a straight line segment to a 1-manifold has no effect on the topology $\overset{+}{\underset{-}{|}}$ must map to the identity map in Alg_k^2 which is the bimodule ${}_AA_A$:

$$ Z\left(\overset{+}{\underset{-}{|}} \right) = {}_AA_A. $$

[5]This means that A-action on ${}_AM_B$ and B-action on ${}_AM_B$ are commutative.

[6]A 2-framing of a smooth l-manifold M, $l \le 2$, is an isomorphism of the vector bundle $\mathbb{R}^{2-l} \oplus TM$ with the trivial bundle $M \times \mathbb{R}^2$.

[7]The source and target is determined by the choice of isomorphism $\mathbb{R}^{2-k} \oplus TM|_B \cong \mathbb{R}^{2-k+1} \oplus TB$ from M onto part of its boundary B.

Also

$$Z\left(\underset{+\sqcup-}{\bigcap}\right) = {}_{A\otimes A^{op}}A$$
$$Z\left(\overset{-\sqcup+}{\bigcup}\right) = A_{A\otimes A^{op}}$$

so gluing them together gives

$$Z(S^1) = A \otimes_{A\otimes A^{op}} A = \frac{A}{[A,A]}$$

as a vector space. We can define a trace on $\frac{A}{[A,A]}$:

$$tr' = Z\left(\mathrm{D}\right) : \frac{A}{[A,A]} \to k.$$

Thus we can define a trace on A by factoring through $\frac{A}{[A,A]}$:

$$tr : A \to \frac{A}{[A,A]} \xrightarrow{tr'} k.$$

Then the pairing $A \otimes A \to k$, defined by $(a,b) \mapsto tr(a \cdot b)$ for all $a, b \in A$, is a Frobenius form and thus A is a Frobenius algebra. To see that $\left(\frac{A}{[A]}\right)^*$ is the centre you prove that the map

$$z \mapsto \big(tr(z \cdot \_) : A \to k\big)$$

is an isomorphism $Z(A) \cong (\frac{A}{[A]})^*$ ($Z(A)$ is the centre of A) using the associativity property of the Frobenius form. Finally to show the Frobenius algebra A is finite dimensional is equivalent to showing it is separable.

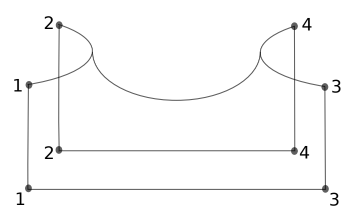


Figure 10: Saddle

To do this we use the map

$$Z(\text{saddle})_1A_2 \otimes {}_3A_4 \to {}_1A_3 \otimes {}_2A_4.$$

The element e which is used to split the multiplication map $m : A \otimes A \to A$ into $A \otimes A = A' \oplus A''$ is defined as

$$e := Z(\text{saddle})(1 \otimes 1) \in A \otimes A.^8$$

[8] e describes how the points (copies of A) are permuted