<u>Origins of topological field theory</u>

First: <u>quantum</u> field theory

Setup: Have some manifold (typically a 4-manifold M^4, equipped w/ a metric w/ signature $(+++,-)$ ("space-time"), or sometimes M^3, equipped w/ a Riemannian metric, or else a surface $\Sigma = M^2$, w/ a conformal structure.

Study some space of "fields" on M. That is, we have some natural bundles

$$\begin{array}{c} \mathcal{E} \\ \downarrow \\ M \end{array}, \quad \text{and we wish to study sections } \Gamma(M, \Omega^* \mathcal{E})$$

In physical situations, there is a Lagrangian

$$\mathcal{L}(\varphi) = f\left(\varphi, \frac{\partial \varphi}{\partial x_i}, \frac{\partial^2 \varphi}{\partial x_i \partial x_j}, \quad \right), \quad f \in \mathbb{C}[z_1, \ldots]$$

and the main object of study is

$$Z = \int_{\varphi \in \Gamma(M, \Omega^* \mathcal{E})} e^{\pi i \int_M \mathcal{L}(\varphi)} \, d\varphi$$

"the partition function", and

$\cancel{Z(\varphi, \psi}$

$\cancel{Z(\psi_1, \psi_2) = \int_\varphi}$

This is the <u>measure</u> against one compares correlations of fields ψ_1, ψ_2, i.e. the likelihood of observing ψ_2 given ψ_1 and vice versa.

These integrals are ill-posed in general, not only divergent but just undefined because $d\varphi$ does not exist as stated.

QFT addresses this in various ways to still get meaningful physical quantities out.

The partition function Z depends, a priori, on the extra data on M, such as a metric. However, in many field theories, esp. "super-symmetric" ones, this dependence is trivial, and the field theory is said to be topological.

Examples: Chern-Simons functional on a 3-manifold.

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Yang Mills functional on a 4-manifold.

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In the 1980's, Atiyah & Segal realized that the formal properties of the (ill-defined) partition function may nevertheless lead to to interesting topological invariants.

~~An invariant of manifolds~~

In particular, they were interested in manifold invariants which exploited the manifold nature, not just the topological space.

The most basic properties are ~~that~~ those of boundary conditions and compatibility w/ cobordisms.

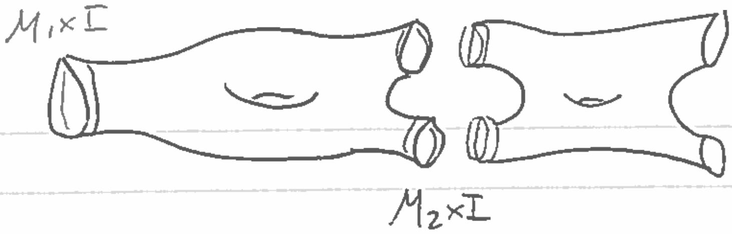
~~Defn~~

1) Given submanifolds $M_1, M_2 \subseteq \partial M$ and fields supported in some collar nbhd $M_1 \times I \subseteq M$, $M_2 \times I \subseteq M$, we can define

$$Z(\varphi_1, \varphi_2) = \int_{\substack{\varphi \in \Gamma(M, \Omega^* \Sigma) \\ \varphi|_{M_1 \times I} = \varphi_1 \\ \varphi|_{M_2 \times I} = \varphi_2}} e^{i\pi \int_M \mathcal{L}(\varphi)} \, d\varphi.$$

This "defines" a map:

$$Z : \mathcal{F}(M_1 \times I) \longrightarrow \mathcal{F}(M_2 \times I).$$



2) These maps "compose" under ~~cobord~~ collar gluing. This is "clear" from the integral formulation.

3) $Z(M \sqcup N) = Z(M) \cdot Z(N)$
$Z(M_1 \sqcup M_2) = Z(M_1) \otimes Z(M_2)$, $M_1 \sqcup M_2 \subseteq \partial M$

$\rightsquigarrow$ Atiyah - Segal studied the cobordism (1-)category.

$Cob^1(n)$: The cobordism 1-category has

Objects = (n-1)-dimensional, compact, oriented (closed) manifolds.

Morphisms = oriented cobordisms M w/
$Hom(M_1, M_2) = \partial M = M_1^+ \sqcup M_2^-$ / diffeomorphism.

Remarks: 1) Orientations needed to determine "source" + "target" of cobordisms.

2) Morphisms are equivalence classes, which is needed for associativity to hold on the nose.

3) Later we will regret that decision, and revisit it using "higher categories".

Atiyah-Segal's TFT:

An n-dim$^{\underline{l}}$ TQFT is a symmetric monoidal functor,

$$Z : \mathrm{Cob}^1(n)^{\sqcup} \longrightarrow \mathrm{Vect}_k^{\otimes}$$

vector spaces w/ "usual" tensor product

This def$^{\underline{n}}$ is already very interesting but we will extend it in various ways.

Given $k = 1, 2, \dots, \infty$, we will see a natural category, and ~~define~~ study functors

$$Z : \mathrm{Cob}^k(n)^{\sqcup} \longrightarrow \mathcal{C}^{,\otimes}$$

some natural higher categories

e.g. $\mathcal{C}^{,\otimes} = \mathrm{Alg}^2_k = \begin{cases} k\text{-algebras as obj.} \\ A\text{-}B\text{-bimodules as mor.} \\ \text{bimodule maps as 2-mor} \end{cases}$

$\mathcal{C}^{,\otimes} = \mathrm{Alg}^2_{\mathrm{Vect}_k} = \begin{cases} \text{tensor categories as obj.} \\ \text{bimodule categories as 1-mor} \\ \text{bimodule functors as 2-mor} \\ \text{natural transformations as 3-mor} \end{cases}$

$\mathcal{C}^{,\otimes}$ = braided tensor categories,
modular tensor categories

$\vdots$

Will see the <u>cobordism hypothesis</u>:

Roughly, can find "generators" and "relations" presentations for $\mathrm{Cob}^k(n)$, and hence classify ~~obj~~ TFT's in terms of special objects in $\mathcal{C}^{\otimes}$ which are called (fully) dualizable.

Cobordism hypothesis:

- $\mathrm{Cob}^{\infty}(n)$ is generated by a single object, the n-disc. Hence a fully extended TFT is completely determined by $Z(\mathbb{R}^n) \in \mathcal{C}^{\otimes}$!

- $Z(\mathbb{R}^n)$ must be fully dualizable, which asserts $\exists$ of many adjoints and duals in $\mathcal{C}^{\otimes}$ and is essentially a strong finiteness condition.

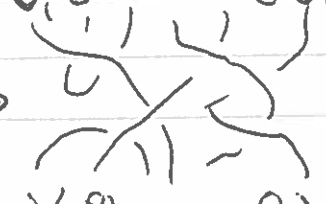
We will see the Reshetikhin-Turaev theory.

Given a simple Lie algebra, e.g. $\mathfrak{g} = \mathfrak{sl}_2$, the universal enveloping algebra $\mathcal{U} = \mathcal{U}_q(\mathfrak{sl}_2)$ has a <u>braided</u> or E_2-structure on its representations.

$$V, W \in \mathrm{Rep}\,\mathcal{U} \rightsquigarrow V \otimes W$$

$V_1 \otimes \cdots \otimes V_m$

Some tangle



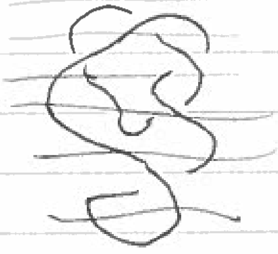
$\rightsquigarrow$ A morphism $V_1 \otimes \cdots V_n \longrightarrow V_{\sigma(1)} \otimes \cdots \otimes V_{\sigma(n)}$

$V_1 \otimes \cdots \otimes V_n$

Elementary, but very deep, to see that this defines "knot invariants" and "braid invariants"

i.e. given $S^1 \hookrightarrow K^3$, we can think of it as $f_K \in \mathrm{Hom}(\mathbb{1}, \mathbb{1})$

$\mathbb{1}$



$\mathbb{1}$

$\rightsquigarrow$ Jones polynomial.

Much less elementary (but not too bad)

This actually defines canonical 3-mfld invariants

$$Z: (M^3) \longrightarrow \mathbb{C},$$

which define a functor

$$Z: \mathrm{Cob}^2(3) \longrightarrow \text{Modular tensor categories (in particular, braided)}$$

This is called a (3,2,1)-dimensional TFT, the Witten-Reshetikhin-Turaev TFT.

Recently, Bartlett, Douglas, Schommer-Pries proved the corresponding cobordism hypothesis for such things.

Recently an important tool for constructing and classifying TFT's appeared, called <u>factorization</u> <u>homology</u>.

Factorization homology returns to the dream, of writing quantities as simple integrals,

$$Z(M) = \int_M \_\_\_\_ \, dm$$

but now instead of integrating <u>numbers</u>, or even <u>vector spaces</u>, we want to integrate <u>higher structures</u> like ~~mono~~ tensor categories, braided tensor categories, categories of <u>quasi-coherent sheaves</u>, <u>mapping spaces</u>,

anything which is defined locally on M, and satisfies a certain "factorization" property similar to that in QFT. w.r.t. disj union.