Cremona group

1. Intro to birational geometry.

everything smooth $/\mathbb{C}$.

Def $X\subset \mathbb{P}^n$, f is a rat function on X if

$\exists f_x = \dfrac{P_N(x_0,\dots,x_n)}{Q_N(x_0,\dots,x_n)}$ rat function s.t. $f = f_x|_X$

Def $g: X \dashrightarrow Y\subset \mathbb{P}^n$ is a rat map if there are $g_0, \dots, g_n$ rat f. s.t. $g(P)=(g_0(P):\dots:g_n(P))$ and Y is a closure of $\operatorname{Im} g$

Def g is birational if there is inverse map. ($\sim$ isomorphism on open dense subset)

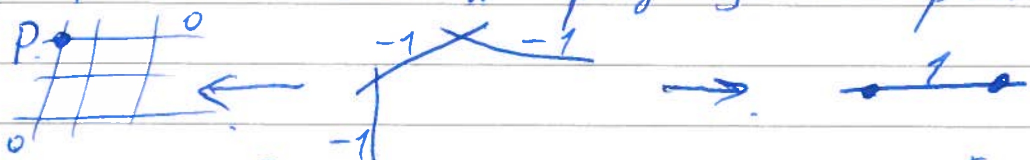
Note if the map is defined everywhere, we say it is regular.

Examples 1) blow up of a point: $\{xu=yv\}\subset \mathbb{A}^2_{x,y}\times\mathbb{P}^1_{u:v} \xrightarrow{f} \mathbb{A}^2_{x,y}$.
1-1 everywhere except $x=y=0$, where preimage is all $(u:v)$, that is $\mathbb{P}^1_{u:v}$. We glue tangent directions instead of point.

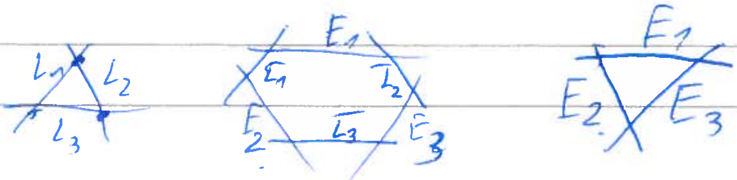
Exercise: convince yourself that a blow up of a point on a disc is a Moebius strip.

Th $C\subset X$ - smooth (Castelnuovo)
C can be contracted to a nonsing point $\Leftrightarrow$ $C^2=-1$, $C\simeq\mathbb{P}^1$

2) $\mathbb{P}^1\times\mathbb{P}^1 \simeq Q\subset \mathbb{P}^3$ project from a point



3) $\mathbb{P}^2 \overset{\sigma}{\dashrightarrow} \mathbb{P}^2$
$(x,y,z)\mapsto (yz:xz:xy)$



F.

Def $Cr_n(\mathbb{C}) = Cr_n$ is a group $Bir(\mathbb{P}^n)$ of birational selfmaps of $\mathbb{P}^n$.

Ex 1) $Cr_1 = PGL_2(\mathbb{C})$

2) Cr_2 is generated by $PGL_3(\mathbb{C})$ and σ

Let us prove 2)

Suppose $f \in Cr_2$, the $f = (f_1 : f_2 : f_3)$, where $f_i \in \mathbb{C}[x,y,z]_n$

$\deg f := n$

Consider $\mathcal{L} := \{\{c_1 f_1 + c_2 f_2 + c_3 f_3 = 0\}\}$. $\mathcal{L}$ is a space of curves on $\mathbb{P}^2$, in fact $\mathcal{L} \simeq \mathbb{P}^2$.
Generic $P \in \mathbb{P}^2$ defines a line $c_1 f_1(P) + \dots = 0$ on $\mathcal{L}^\vee$
Thus $f : \mathbb{P}^2 \dashrightarrow \mathcal{L}^\vee$, f is undefined at base points of $\mathcal{L}$. (Example σ)

Lemma $\mathcal{L}_f$ has $3(\deg f - 1)$ base points

□ proof is by induction, I will only do 1 step.

$f \notin PGL \Rightarrow \exists$ base point P. Blow up P, then for gen $C \in \mathcal{L}$

$$\begin{array}{ccc} & S & \\ {\scriptstyle\varphi}\swarrow & & \searrow{\scriptstyle\psi} \\ \mathbb{P}^2 & & \mathcal{L}^\vee \end{array}$$

$$C_S = \varphi^* C - \overset{\text{mult.}}{m}\,\overset{\text{exc line}}{E}$$

$$K_S = \varphi^*(-3L) + E$$

can't happen for 1 blow up but imagine it did.

(define it here)

On the other hand $C_S = \psi^*(L_{\mathcal{L}^\vee})$, and $K_S = \psi^* K_{\mathcal{L}^\vee} + E_{\mathcal{L}^\vee}$

$$-3n = C_S \cdot K_S - m = L_{\mathcal{L}^\vee} \cdot K_{\mathcal{L}^\vee} + \psi^*(\dots) \cdot E_{\mathcal{L}^\vee} - m = -3 - m$$

Now take 3 points on $\mathbb{P}^2$ and do σ at them (change coord and σ) then you get $f_? = f' \circ \sigma_1$, $\deg f' < \deg f$, proceed by induction

Deceit! Where?

Things are hopeless in dimension 3.

Finite Subgroups

Very hard to classify. How far can they be from abelian?

PGL is Jordan: $\forall G \overset{fin}{\subset} PGL_n$ $\exists A \triangleleft G$ s.t. $[G:A] \le I(n)$

Cr is also Jordan (worst things are $A_5 \times \dots \times A_5$ on $\mathbb{P}^1 \times \dots \times \mathbb{P}^1$)

~~Probably~~ Can classify simple groups

Cr_1 A_5, Cr_2 add $PSL_2(7)$, A_6, Cr_3: $SL_2(8)$, $SU_2(4)$

Suppose $G \not\subset PGL_3$, then there is a regularization that is rat. X, $G \subset Aut(X)$. ($U \subset \mathbb{P}^2$, $G \curvearrowright U$, $Y := U/G$, norm Y in $k(U)$)

X may be singular $\to$ resolve sing (G-equivariantly) and run G-MMP to get 1) G-minimal conic bundle $f: S \to \mathbb{P}^1$

f is G-equivariant G-equiv curves$/\sim$ / fibers $= \mathbb{Z}$

2) G-minimal dP surface S (G-equiv curves$/\sim$ $= \mathbb{Z}$)

In case 1 we have action of G on base or fibers $\Rightarrow$

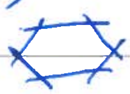
$\Rightarrow G = A_5$ (if G is simple

Case 2

$K_S^2 = 9$ $\mathbb{P}^2$ repr th.

$K_S^2 = 8$ $\mathbb{F}_1$ or $\mathbb{P}^1 \times \mathbb{P}^1$ not G-minimal.

$K_S^2 = 7$  not G-min.

$K_S^2 = 6$  a map to D_6

$K_S^2 = 5$ $Gr(2,5)$ exercise for audience

$K_S^2 = 4$ $Q_1 \cap Q_2$ $G \curvearrowright \langle Q_1, Q_2 \rangle = \mathbb{P}^1$ 5 deg fibers

$|-K_S|$ stab is a cyclic group of order 12.

$K_S^2 = 3$ $|-K_S|$ unique with action not minimal.

$K_S^2 = 2$ $S \dashrightarrow \mathbb{P}^2 \supset Q_4$ Klein.

$K_S^2 = 1$

3-dim $\Rightarrow$ rationality problems

conjugacy if time permits