Comes from Coulomb's law

Gauss's law:

$\varepsilon_0 = 1, \mu_0 = 1, c = 1$

$$\oint_{\partial\Omega} \underline{E} \cdot d\underline{S} = \int_{\Omega} \rho \, dV = Q$$ - electric charge

Electric flux

$\underline{F} = q\underline{E}$

$$\oint_{\partial\Omega} \underline{B} \cdot d\underline{S} = 0$$ - no magnetic charges

Faraday's Law of induction:

$$\int_{\partial\Sigma} \underline{E} \cdot d\underline{l} = -\frac{d}{dt} \int_{\Sigma} B \cdot dS$$

Voltage

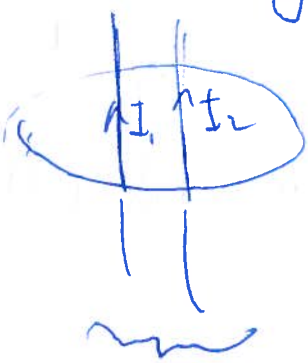
change of magnetic flux

Ampere's Law

$$\int_{\partial\Sigma} B \cdot d\underline{l} = \int_{\Sigma} \underline{J} \, d\underline{S} + \frac{d}{dt} \int_{\Sigma} \underline{E} \, d\underline{S}$$

electric current density

$$\nabla \cdot J = -\frac{\partial \rho}{\partial t}$$




$I_{encl}$

Apply divergence and Kelvin-Stokes theorems

$$\nabla \cdot \underline{E} = \rho$$
$$\nabla \cdot \underline{B} = 0$$
$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$
$$\nabla \times \underline{B} = \underline{J} + \frac{\partial \underline{E}}{\partial t}$$

$$\nabla \cdot (\nabla \times \underline{B}) = 0$$

This looks a bit simpler but we can go further.

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Consider $\mathbb{R}^n$ with metric $\eta = -d\underline{t}^2 + d\underline{x}^2$ $\begin{pmatrix} - & & & \\ & - & & \\ & & + & \\ & & & + \end{pmatrix}$

signature $(-,+) = (t, n-t)^T$

assume oriented space, so we have a volume form

$$\frac{1}{n!}\sum_{a_1,\dots,a_n} \varepsilon_{a_1\dots a_n} dx^{a_1} \wedge \dots \wedge dx^{a_n}.$$

$$\lambda \wedge M = (\star\lambda, M)\,\varepsilon$$

$\lambda$: p form, $M$: n-p form, $\varepsilon$: top form

$$(\star\lambda)_{a_1,\dots,a_{n-p}} = \frac{(-1)^t}{p!}\,\varepsilon^{b_1\dots b_{n-p}}{}_{a_1\dots a_q}\,\lambda_{b_1\dots b_p}.$$

(Lemma: $\lambda$ p-form, then $\star\star\lambda = (-1)^t(-1)^{p(n-p)}\lambda$)

Let us work in $\mathbb{R}^4 = \mathbb{R}^{3,1}$, $t=1$,

~~Let~~. $\varepsilon = dt \wedge dx \wedge dy \wedge dz$

Let $F$ be 2-form.

~~(Then $**F = -F$)~~

$$F = F_{0i}\, dt \wedge dx^i + \frac{1}{2} F_{ij}\, dx^i \wedge dx^j$$

$F$ has 6 independent components. Let us call them

$$E_i = -F_{0i}, \quad B_i = \frac{1}{2}\varepsilon_{ijk} F_{jk}$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

$$*F_{\mu\nu} = \begin{pmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & E_3 & -E_2 \\ -B_2 & -E_3 & 0 & E_1 \\ -B_3 & E_2 & -E_1 & 0 \end{pmatrix}$$

---

We ~~need~~ need one extra assumption. Suppose $F$ is exact. Then $F = dA$, $A = -\phi\, dt + A_i\, dx^i$

$$\Rightarrow dF = d^2 A = 0$$

$$\Rightarrow \quad \underline{E} = -\frac{\partial \underline{A}}{\partial t} - \nabla\phi \qquad \nabla \times \underline{E} = -\frac{\partial(\nabla \times \underline{A})}{\partial t} - \nabla \times \nabla\phi = -\frac{\partial B}{\partial t}$$

$$\underline{B} = \nabla \times \underline{A} \qquad \Rightarrow \nabla \cdot \underline{B} = \nabla \cdot (\nabla \times \underline{A}) = 0$$

The other two equations come from

$$d * F = * J$$

Both can be derived by extremizing $\int (F \wedge *F)$

Note that $A$ is not unique

$$F = dA \Rightarrow \quad A \mapsto A' = A + dC \text{ then } F \mapsto F$$

There are many different $A$'s corresponding to the same physical system (transformations between them - gauge transf.)

Generalization: Yang-Mills

Let $G$ be a Lie group, $\mathfrak{g}$ it's Lie algebra

$A$ - $\mathfrak{g}$ valued 1-form

$F = dA + A \wedge A$ 2-form, $F_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b]$ = ~~[...]~~

Note that $F_{ab} = [D_a, D_b]$ where $D := d + A$ ← covariant derivative

gauge transf: $A' = gAg^{-1} - dg\, g$, $F' = gFg^{-1}$.

Equations $DF = 0$, $D * F = J$.

$G = U(1) \rightarrow$ Electromagnetism

$\int \mathrm{Tr}(F \wedge * F)$

Principle ~~fibre~~ bundle is ~~[...]~~ $\{E, B, \pi, G\}$

$E$, $B$ are manifolds, smooth surjection $\pi : E \to B$

$G$ acts on itself by left translations; $E$ – total space, $B$ – base space (physical)

Define lie algebra valued connection $\omega$ on $E$ s.t. it's component on fibre is $\gamma^{-1} d\gamma$, $\gamma : B \to G$

$$\omega = \gamma^{-1} A \gamma + \gamma^{-1} d\gamma \quad \text{(locally)}$$

$A$ – 1-form on $B$

A section of a bundle $\pi : E \to B$ is a map $s : B \to E$ s.t. $\pi \circ s = id_B$

Curvature $\Omega = d\omega + \omega \wedge \omega = \gamma^{-1} F \gamma$; ~~F...B~~

$F$ – 2 form on $B$



Exercise: show that $F = dA + A \wedge A$

# Curved space

Suppose we have a metric $g_{ab}$.

Orthonormal basis of vector fields $\{e^a_\mu\}$

s.t. $\quad g_{ab}\, e^a_\mu e^b_\nu = \eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

define connection 1-form

$$(\omega^\mu{}_\nu)_a = e^\mu_b \nabla_a e^b_\nu$$

↑ Levi-Civita

Lemma: $\quad de^\mu = -\omega^\mu{}_\nu \wedge e^\nu$

$$\Leftrightarrow (de^\mu)_{\nu\rho} = 2(\omega^\mu{}_{[\nu})_{\rho]}$$

Curvature 2-form

$$\Omega^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu$$

$$= \frac{1}{2} R^\mu{}_{\nu\rho\sigma}\, e^\rho \wedge e^\sigma \quad \text{(exercise)}$$

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Example $\quad ds^2 = -f^2 dt^2 + \frac{1}{f^2} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$

$$f(r) = \sqrt{1 - \frac{2M}{r}}, \quad M > 0$$

$e^0 = f\,dt, \quad e^1 = f^{-1} dr, \quad e^2 = r\,d\theta, \quad e^3 = r\sin\theta\, d\phi$

$de^0 = f' e^1 \wedge e^0, \quad de^1 = 0, \quad de^2 = dr \wedge d\theta = \frac{f}{r} e^1 \wedge e^2$

$de^3 = \sin\theta\, dr \wedge d\phi + r\cos\theta\, d\theta \wedge d\phi = \frac{f}{r} e^1 \wedge e^3 + \frac{1}{3}\cot\theta\, e^2 \wedge e^3$

read-off $\omega^0{}_1 = f' e^0$

$\omega^2{}_1 = + \frac{f}{r} e^2$

$\omega^3{}_1 = \frac{f}{r} e^3 \qquad \omega^3{}_2 = \frac{1}{r} \cot\theta \, e^3$

$$d\omega^0{}_1 = f' de^0 + f'' dr \wedge e^0 = f'^2 e^1 \wedge e^0 + f f'' e^1 \wedge e^0$$

$$\omega^0{}_\rho \wedge \omega^\rho{}_1 = 0$$

$$\Rightarrow \Theta_{01} = \cancel{\Theta^0{}_1} = (f f'' + f'^2)\, e^0 \wedge e^1$$

$$R_{0\rho 1\rho} \longrightarrow R_{0101} = -R_{0110} = (f f'' + f'^2) = \frac{1}{2}(f^2)'' = -\frac{2M}{r^3}.$$