Generalised Jucys-Murphy operators and the cactus group

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February 16, 2015

Combinatorics
- The RSK correspondence
- The Bonnafé-Rouquier conjecture
- Schützenberger's involution

Representation theory
- Crystals
- The cactus group

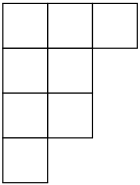
Geometry
- The Gaudin Hamiltonians
- Compactifying and calculating the monodromy

Partitions

A *partition* is a non-increasing sequence of positive integers.

$(3, 2, 2, 1)$ is a partition of 8.

We display partitions using *Young diagrams*.



n will be the number of boxes, r is a natural number.

Tableaux

We can fill these diagrams to create *Young tableaux*, which are

- *semistandard* if filled using $1, 2, \ldots, r$, with weakly increasing rows, strictly increasing columns.
- *standard* if filled using $1, 2, \ldots, n$ *bijectively*, with strictly increasing rows and columns.

2	2	4
3	5	
6	6	
7		

semistandard

1	3	4
2	6	
5	7	
8		

standard

Insertion

Given a tableau T and a number x, we can *insert* x into T and produce a new tableau $(T \leftarrow x)$.

2	2	4	$\longleftarrow 3$
3	5		
6	6		
7			

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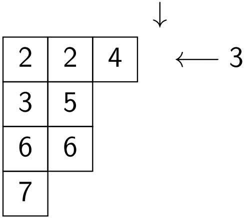

Insertion

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		$\downarrow$	
2	2	4	$\longleftarrow 3$
3	5		
6	6		
7			

Insertion

Given a tableau T and a number x, we can *insert* x into T and produce a new tableau $(T \leftarrow x)$.

2	2	3
3	5	
6	6	
7		

4

Insertion

Given a tableau T and a number x, we can *insert* x into T and produce a new tableau $(T \leftarrow x)$.

2	2	3	
3	5		$\longleftarrow$ 4
6	6		
7			

Insertion

Given a tableau T and a number x, we can *insert* x into T and produce a new tableau $(T \leftarrow x)$.

2	2	3
3	4	
6	6	
7		

5

Insertion

Given a tableau T and a number x, we can *insert* x into T and produce a new tableau $(T \leftarrow x)$.

2	2	3
3	4	
6	6	
7		

$\longleftarrow 5$

Insertion

Given a tableau T and a number x, we can *insert* x into T and produce a new tableau $(T \leftarrow x)$.

2	2	3
3	4	
5	6	
7		

6

Insertion

Given a tableau T and a number x, we can *insert* x into T and produce a new tableau $(T \leftarrow x)$.

2	2	3
3	4	
5	6	
7		$\longleftarrow 6$

Insertion

Given a tableau T and a number x, we can *insert* x into T and produce a new tableau $(T \leftarrow x)$.

2	2	3
3	4	
5	6	
6		

7

Insertion

Given a tableau T and a number x, we can *insert* x into T and produce a new tableau $(T \leftarrow x)$.

2	2	3
3	4	
5	6	
6		

$\longleftarrow 7$

Insertion

Given a tableau T and a number x, we can *insert* x into T and produce a new tableau $(T \leftarrow x)$.

2	2	3
3	4	
5	6	
6		
7		

The Robinson-Schensted-Knuth correspondence

To a word $w = x_1 x_2 \cdots x_n$ we associate the pair of tableaux

$$(P, Q)$$

$$P := ((\ldots((\emptyset \leftarrow x_1) \leftarrow x_2)\ldots) \leftarrow x_n)$$
$$Q := sh(P_1) \subset sh(P_2) \subset \ldots sh(P_n)$$

where $P_i = ((\ldots((\emptyset \leftarrow x_1) \leftarrow x_2)\ldots) \leftarrow x_i)$.

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Theorem (Robinson, Schensted, Knuth)

This defines a bijection

$$\text{words}(n) \longrightarrow \bigsqcup_{\lambda \vdash n} \text{SSYT}(\lambda) \times \text{SYT}(\lambda)$$

An example of RSK

Take $w = 26327653$.

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$\boxed{2}$,

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$\boxed{2}$, $\boxed{2}\boxed{6}$,

An example of RSK

Take $w = 26327653$.

$\boxed{2}$, $\boxed{2 \mid 6}$,

An example of RSK

Take $w = 26327653$.

$$\begin{array}{|c|}\hline 2 \\ \hline \end{array}, \quad \begin{array}{|c|c|}\hline 2 & 6 \\ \hline \end{array}, \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 6 \\ \cline{1-1} \end{array},$$

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$$\begin{array}{|c|}\hline 2\\\hline\end{array}\,, \quad \begin{array}{|c|c|}\hline 2&6\\\hline\end{array}\,, \quad \begin{array}{|c|c|}\hline 2&3\\\hline 6\\\cline{1-1}\end{array}\,, \quad \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1} 6\\\cline{1-1}\end{array}\,, \quad \begin{array}{|c|c|c|}\hline 2&2&7\\\hline 3\\\cline{1-1} 6\\\cline{1-1}\end{array}\,,$$

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$$\begin{array}{|c|}\hline 2 \\ \hline \end{array}, \quad \begin{array}{|c|c|}\hline 2 & 6 \\ \hline \end{array}, \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 6 \\ \cline{1-1} \end{array}, \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array}, \quad \begin{array}{|c|c|c|}\hline 2 & 2 & 7 \\ \hline 3 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array},$$

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$$\begin{ytableau} 2 \end{ytableau}, \quad \begin{ytableau} 2 & 6 \end{ytableau}, \quad \begin{ytableau} 2 & 3 \\ 6 \end{ytableau}, \quad \begin{ytableau} 2 & 2 \\ 3 \\ 6 \end{ytableau}, \quad \begin{ytableau} 2 & 2 & 7 \\ 3 \\ 6 \end{ytableau}, \quad \begin{ytableau} 2 & 2 & 6 \\ 3 & 7 \\ 6 \end{ytableau}, \quad \begin{ytableau} 2 & 2 & 5 \\ 3 & 6 \\ 6 & 7 \end{ytableau},$$

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$$\begin{ytableau} 2 \end{ytableau}, \quad \begin{ytableau} 2 & 6 \end{ytableau}, \quad \begin{ytableau} 2 & 3 \\ 6 \end{ytableau}, \quad \begin{ytableau} 2 & 2 \\ 3 \\ 6 \end{ytableau}, \quad \begin{ytableau} 2 & 2 & 7 \\ 3 \\ 6 \end{ytableau}, \quad \begin{ytableau} 2 & 2 & 6 \\ 3 & 7 \\ 6 \end{ytableau}, \quad \begin{ytableau} 2 & 2 & 5 \\ 3 & 6 \\ 6 & 7 \end{ytableau}, \quad \begin{ytableau} 2 & 2 & 3 \\ 3 & 5 \\ 6 & 6 \\ 7 \end{ytableau} = P$$

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$$\begin{ytableau} 2 \end{ytableau}, \quad \begin{ytableau} 2 & 6 \end{ytableau}, \quad \begin{ytableau} 2 & 3 \\ 6 \end{ytableau}, \quad \begin{ytableau} 2 & 2 \\ 3 \\ 6 \end{ytableau}, \quad \begin{ytableau} 2 & 2 & 7 \\ 3 \\ 6 \end{ytableau}, \quad \begin{ytableau} 2 & 2 & 6 \\ 3 & 7 \\ 6 \end{ytableau}, \quad \begin{ytableau} 2 & 2 & 5 \\ 3 & 6 \\ 6 & 7 \end{ytableau}, \quad \begin{ytableau} 2 & 2 & 3 \\ 3 & 5 \\ 6 & 6 \\ 7 \end{ytableau} = P$$

$$\ydiagram{1}, \quad \ydiagram{2}, \quad \ydiagram{2,1}, \quad \ydiagram{2,1,1}, \quad \ydiagram{3,1,1}, \quad \ydiagram{3,2,1}, \quad \ydiagram{3,2,2}, \quad \ydiagram{3,2,2,1}$$

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Take $w = 26327653$.

$$\begin{array}{|c|}\hline 2\\\hline\end{array}, \quad \begin{array}{|c|c|}\hline 2&6\\\hline\end{array}, \quad \begin{array}{|c|c|}\hline 2&3\\\hline 6\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1} 6\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 2&2&7\\\hline 3\\\cline{1-1} 6\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 2&2&6\\\hline 3&7\\\cline{1-2} 6\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 2&2&5\\\hline 3&6\\\cline{1-2} 6&7\\\cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|}\hline 2&2&3\\\hline 3&5\\\cline{1-2} 6&6\\\cline{1-2} 7\\\cline{1-1}\end{array} = P$$

$$\begin{array}{|c|}\hline \\\hline\end{array}, \quad \begin{array}{|c|c|}\hline &\\\hline\end{array}, \quad \begin{array}{|c|c|}\hline &\\\hline \\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline &\\\hline \\\cline{1-1} \\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline &&\\\hline \\\cline{1-1} \\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline &&\\\hline &\\\cline{1-2} \\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline &&\\\hline &\\\cline{1-2} &\\\cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&6\\\cline{1-2} 4&7\\\cline{1-2} 8\\\cline{1-1}\end{array} = Q$$

Kazhdan-Lusztig cells

We can interpret a permutation $\sigma \in S_n$ as a word of length n.

$$\text{e.g. } (1,2) = 2134 \cdots n$$

Definition

We say two permutations σ and τ belong to the same *cell* if

$$P(\sigma) = P(\tau).$$

These are the *Kazhdan-Lusztig cells*. They are important in representation theory. They are defined for any *Coxeter group*.

The Bonnafé-Rouquier conjecture

Conjecture (Bonnafé-Rouquier 2013)

Let W be a Coxeter group. There is a group G acting on W whose orbits are exactly the Kazhdan-Lusztig cells of W.

The group G is defined using the Galois theory of the *Calogero-Moser space* associated to W. The definition works for any *complex reflection group*.

Schützenberger's involution

There is an involution ξ on semistandard tableaux. It is defined in the following way (suppose $r = 9$).

2	2	4
3	5	
6	6	
7		

Note: $k^* = r + 1 - k$ (or $n + 1 - k$ if restricting to standard tableaux).

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3	5	
6	6	
7		

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3	4	2^*
$\bullet$	5	
6	6	
7		

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3	4	2^*
5	6	
6	•	
7		

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<table>
<tr><td>3</td><td>4</td><td>2*</td></tr>
<tr><td>5</td><td>6</td><td></td></tr>
<tr><td>6</td><td>2*</td><td></td></tr>
<tr><td>7</td><td></td><td></td></tr>
</table>

Note: $k^* = r + 1 - k$ (or $n + 1 - k$ if restricting to standard tableaux).

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$\bullet$	4	2^*
5	6	
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5	6	
6	2^*	
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Schützenberger's involution

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4	6	2^*
5	$\bullet$	
6	2^*	
7		

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Schützenberger's involution

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4	6	2^*
5	3^*	
6	2^*	
7		

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$\bullet$	6	2^*
5	3^*	
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7		

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•		

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5	6	2^*
6	3^*	
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4^*		

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$\bullet$	6	2^*
6	3^*	
7	2^*	
4^*		

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6	6	2^*
$\bullet$	3^*	
7	2^*	
4^*		

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6	6	2^*
7	3^*	
$\bullet$	2^*	
4^*		

Note: $k^* = r + 1 - k$ (or $n + 1 - k$ if restricting to standard tableaux).

Schützenberger's involution

There is an involution ξ on semistandard tableaux. It is defined in the following way (suppose $r = 9$).

6	6	2^*
7	3^*	
5^*	2^*	
4^*		

Note: $k^* = r + 1 - k$ (or $n + 1 - k$ if restricting to standard tableaux).

Schützenberger's involution

There is an involution ξ on semistandard tableaux. It is defined in the following way (suppose $r = 9$).

7^*	6^*	2^*
6^*	3^*	
5^*	2^*	
4^*		

Note: $k^* = r + 1 - k$ (or $n + 1 - k$ if restricting to standard tableaux).

Schützenberger's involution

There is an involution ξ on semistandard tableaux. It is defined in the following way (suppose $r = 9$).

3	4	8
4	7	
5	8	
6		

Note: $k^* = r + 1 - k$ (or $n + 1 - k$ if restricting to standard tableaux).

Permuting standard tableaux

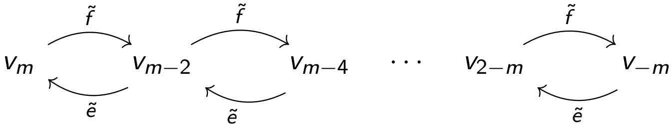
We define $\dot{s}_{1q}$ as the *partial Schützenberger involutions* (perform the Schützenberger involution on the letters $1, 2, \ldots, q$).

$$\dot{s}_{1q} : \mathsf{SYT}(\lambda) \longrightarrow \mathsf{SYT}(\lambda)$$

These operators have been studied by Berenstein and Kirillov.

Crystals for $\mathfrak{sl}_2$

The irreducible $\mathfrak{sl}_2$-modules are classified by the integers. $L(m)$ is the irreducible of dimension $m + 1$. We can draw a diagram:



This is a nice basis! It allows us to the present the action of $\mathfrak{sl}_2$.

$L(1) \otimes L(1) = L(0) \oplus L(2)$

$L(1) = \langle v_1, v_{-1} \rangle$. The tensor basis is

$$v_1 \otimes v_1 \qquad v_1 \otimes v_{-1}$$

$$v_{-1} \otimes v_1 \qquad v_{-1} \otimes v_{-1}$$

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$$v_1 \otimes v_1 \qquad v_1 \otimes v_{-1}$$

$$v_{-1} \otimes v_1 \qquad v_{-1} \otimes v_{-1}$$

However we would prefer the basis

$$L(2) = \begin{cases} v_1 \otimes v_1, \\ f(v_1 \otimes v_{-1}) = v_{-1} \otimes v_1 + \quad v_1 \otimes v_{-1}, \\ f^2(v_1 \otimes v_{-1}) = v_{-1} \otimes v_{-1}, \end{cases}$$

$$L(0) = \Big\{ v_1 \otimes v_{-1} - \quad v_{-1} \otimes v_1$$

$L(1) \otimes L(1) = L(0) \oplus L(2)$ for $U_q(\mathfrak{sl}_2)$

$L(1) = \langle v_1, v_{-1} \rangle$. The tensor basis is

$$\begin{matrix} v_1 \otimes v_1 & v_1 \otimes v_{-1} \\ \\ v_{-1} \otimes v_1 & v_{-1} \otimes v_{-1} \end{matrix}$$

However we would prefer the basis

$$L(2) = \begin{cases} v_1 \otimes v_1, \\ f(v_1 \otimes v_{-1}) = v_{-1} \otimes v_1 + q v_1 \otimes v_{-1}, \\ f^2(v_1 \otimes v_{-1}) = v_{-1} \otimes v_{-1}, \end{cases}$$

$$L(0) = \left\{ v_1 \otimes v_{-1} - q v_{-1} \otimes v_1 \right.$$

$L(1) \otimes L(1) = L(0) \oplus L(2)$ for $U_q(\mathfrak{sl}_2)$ at $q = 0$

$L(1) = \langle v_1, v_{-1} \rangle$. The tensor basis is

$$v_1 \otimes v_1 \qquad v_1 \otimes v_{-1}$$

$$v_{-1} \otimes v_1 \qquad v_{-1} \otimes v_{-1}$$

However we would prefer the basis

$$L(2) = \begin{cases} v_1 \otimes v_1, \\ f(v_1 \otimes v_{-1}) = v_{-1} \otimes v_1 \\ f^2(v_1 \otimes v_{-1}) = v_{-1} \otimes v_{-1}, \end{cases} ,$$

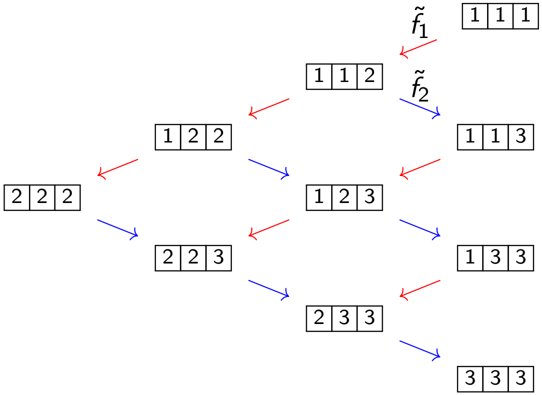
$$L(0) = \Big\{ v_1 \otimes v_{-1}$$

We call this nice basis at $q = 0$ the *crystal*.

Crystals for $\mathfrak{gl}_r$

- Basically, finite dimensional irreducible $\mathfrak{gl}_r$-modules are classified by partitions.
- We label them $L(\lambda)$ and the corresponding crystal $B(\lambda)$.
- A basis is labelled by the semistandard λ-tableaux.

For example, if $\lambda = (3)$, $B(3)$ is



Crystals

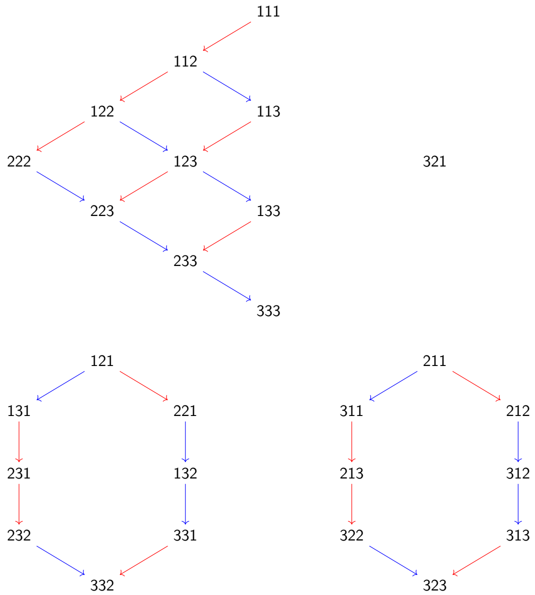
A crystal is a *directed graph* with

- vertices labelled by basis elements (semistandard tableaux)
- arrows indicating the action of $\tilde{f}_i$ and $\tilde{e}_i$

Crystals

- are a schematic representation of the module
- behave well under tensor product
- have well defined weights and highest weight vectors

$L(1)^{\otimes 3} = L(3) \oplus L(2,1)^{\oplus 2} \oplus L(1,1,1)$ for $\mathfrak{gl}_3$



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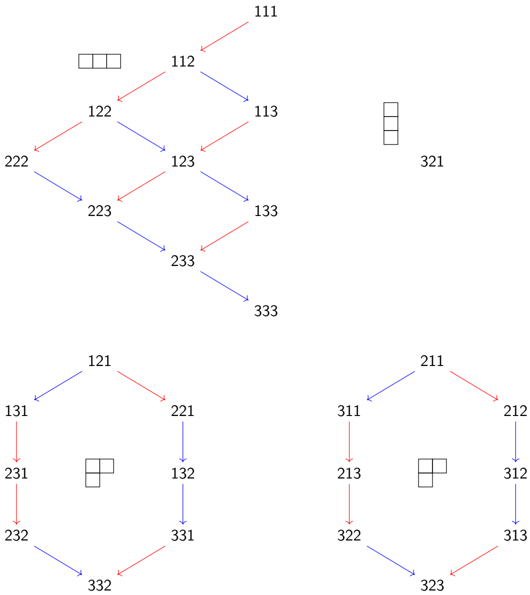

A coboundary structure on crystals

Theorem (Henriques-Kamnitzer 2006)

There are natural isomorphisms $\sigma_{AB} : A \otimes B \longrightarrow B \otimes A$ *satisfying the axioms of a coboundary monoidal category.*

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For a partition λ we define an involution on the $\mathfrak{gl}_r$-crystal $B(\lambda)$ by

$$b \mapsto \xi(b)$$

where $\xi(b)$ is the *Schützenberger involution* of the semistandard λ-tableau $b \in B(\lambda)$.

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Now define

$$\sigma(a \otimes b) = \xi(\xi(b) \otimes \xi(a)).$$

The cactus group

Consider $A_1 \otimes A_2 \otimes \cdots \otimes A_n$. Define s_{pq} as natural isomorphisms reversing the interval $A_p \otimes A_{p+1} \otimes \cdots \otimes A_q$. e.g. s_{25}, $n = 6$

$$s_{25} : A_1 \otimes A_2 \otimes A_3 \otimes A_4 \otimes A_5 \otimes A_6 \longrightarrow A_1 \otimes A_5 \otimes A_4 \otimes A_3 \otimes A_2 \otimes A_6$$

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The cactus group

The isomorphisms s_{pq} obey the following relations

- $s_{pq}^2 = 1$,
- $s_{pq}s_{kl} = s_{kl}s_{pq}$ if $[p, q] \cap [k, l] = \emptyset$, and
- $s_{pq}s_{kl} = s_{uv}s_{pq}$ if $[k, l] \subseteq [p, q]$.

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Definition

The (infinite) group J_n with generators s_{pq} and the above relations is called the *n-flowered cactus group*.

The cactus group acting on $B^{\otimes n}$

We can identify words$(n) = B^{\otimes n}$ and thus $S_n \subset B^{\otimes n}$.

Observation

The orbits of J_n on S_n are exactly the Kazhdan-Lusztig cells.

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Observation

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Proof.

The cactus group acts on $B^{\otimes n} = \text{words}(n)$ by *Knuth moves*. □

Calogero-Moser space

- $q = (q_1, \ldots, q_n) \in \mathbb{C}^n$,
- $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ distinct complex numbers,
- $M = [V^{\otimes n}]_\mu$, a direct sum of Specht modules, $S_\mu^{\oplus N}$.

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The spectrum of the commuting operators

$$H_a(q, z) = \sum_{b=1}^{n} q_b e_{bb}^{(a)} + \sum_{b \neq a} \frac{(a, b)}{z_a - z_b}$$

for $a = 1, \ldots, n$, is organised into a family over

$$\mathbb{C}^n \times (\mathbb{C}^n - \Delta).$$

For $\mu = (1, 1, \ldots, 1)$ this is (basically) *Calogero-Moser space*.

Gaudin Hamiltonians

We are interested the *nilpotent slice* when we set $q = (0, 0, \dots, 0)$. The *Gaudin Hamiltonians*

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act on $M = [V^{\otimes n}]_{\mu}^{\text{sing}}$ (a Specht module!). We can organise the spectrum into a family $\pi : \mathcal{G} \longrightarrow X_n$.

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Theorem (Mukhin-Tarasov-Varchenko)

For generic complex z and all real z, the $H_a(z)$ are diagonalisable and the simultaneous eigenspaces are 1-dimensional.

In particular, π is a branched cover.

The main question

Question

What is the Galois group of π and what are its orbits? A partial answer, conjectured by Brochier-Gordon and Etingof:

Theorem (W.)

There exists a map $J_n \to \mathrm{Gal}(\pi)$ such that the induced action of J_n on $\pi^{-1}(z)$ is the same as the action of J_n on $[B^{\otimes n}]_\mu^{\mathrm{sing}}$.

The AFV theorem

- The $H_a(z)$ generate a fin. dim. algebra over every z. We call this family of algebras G.
- There is a nice compactification of X_n, denoted $\overline{M}_{0,n+1}(\mathbb{C})$.

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The family G extends to a family of commutative subalgebras over $\overline{M}_{0,n+1}(\mathbb{C})$.

Remark

$$\pi_1[\overline{M}_{0,n+1}(\mathbb{R})/S_n] = J_n$$

This suggests we should calculate the monodromy. However, this family is not flat and is not a vector bundle, it drops dimension!

The proof

Theorem (W.)

For generic z there is a J_n-equivariant bijection

$$\pi^{-1}(z) \longrightarrow [B^{\otimes n}]_{\mu}^{\mathrm{sing}}.$$

Proof.

Uses work of Mukhin-Tarasov-Varchenko and Speyer. We identify the action on $\pi^{-1}(z)$ with the action of the *partial Schützenberger involutions* $\dot{s}_{1q}$ on standard tableaux. □

This shows that the action of J_n which produces the cells in S_n is related to Bonnafé and Rouquier's construction and provides strong evidence that their conjecture is true.