1

Brief Introduction to Algebraic K-theory

- K-theory was introduced by Grothendieck in the 1950's as a tool in his reformulation and reproof of the Riemann-Roch Theorem for projective algebraic varieties.

covariant $\quad K : \mathcal{C} \longrightarrow Ab$

category of coherent algebraic sheaves on a fixed proj variety X

category of abelian groups.

- Around 1960, Atiyah + Hirzebruch considered a topological analog (now called (classical) topological K-theory)

contravariant $\quad K_{\mathbb{F}} : \mathcal{C} \longrightarrow Ab$

category of compact Hausdorff topological spaces with base point

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

$K_{\mathbb{R}} =: KO, \quad K_{\mathbb{C}} = KU$

Eilenberg Steenrod

$h_n : \{(X,A)\} \longrightarrow Ab$

$+ \ \partial : H_n(X,A) \to H_{n-1}(A,\emptyset)$

- htpy
- excision
- additivity
- exactness

Indeed, defining
$$KO_n(X) = KO(\Sigma^n X)$$
$$KU_n(X) = KU(\Sigma^n X)$$

the functors KO_n and KU_n were the first examples of a generalised ~~the~~ cohomology theory (in the sense of Eilenberg-Steenrod) and, in some sense, gave birth to modern day homotopy theory.

What are these "K" functors and what do they do?

<u>Def'n</u> Let $\mathcal{C}$ be an additive category i.e.

- Enriched over Ab
- Finite coproducts (+ products) "$\oplus$"

Let $\Phi(\mathcal{C})$ be the set of iso classes. $\oplus \Rightarrow$ commutative monoid.

"Grothendieck group" $K(\mathcal{C}) :=$ group completion of $(\Phi(\mathcal{C}), \oplus)$.

$$= \Phi(\mathcal{C}) \times \Phi(\mathcal{C}) / \sim$$

$$([x],[y]) \sim ([x'],[y']) \text{ iff } \exists z \text{ s.t. } [x] \oplus [y'] \oplus [z] = [x'] \oplus [y] \oplus [z].$$

<u>e.g.</u> $\mathcal{C} = \mathbb{N}$, $[n] = \{n\} \in \Phi(\mathbb{N})$

and $K(\mathbb{N}) = \mathbb{Z}$.

<u>e.g.</u> Fix X pointed compact Hausdorff. $\mathcal{C} = Vect_{\mathbb{F}}(X)$ $\mathbb{F} = \mathbb{R}, \mathbb{C}$

Define. $K_{\mathbb{F}}(X) = K(\mathcal{C})$

<u>FACTS</u> • $E \in Vect_{\mathbb{F}}(X) \Rightarrow \exists F \in Vect_{\mathbb{F}}(X)$ s.t. $E \oplus F \cong \underline{\mathbb{F}}^N$ (N large)

• For any group G There is a space BG that "classifies" principal bundles

$$\left\{ \begin{array}{l} \text{fibre bundles over} \\ X \text{ with fibre automorphism } G \end{array} \right\} \overset{1:1}{\longleftrightarrow} [X, BG] \quad \text{htpy classes of maps}$$

• SES of v.b.s ... iso ... i.e. EULER CHAR = 0 in K

2

e.g. $BGL(n,\mathbb{R}) = Gr(n,\infty) = \varinjlim_k Gr(n, n+k)$ Grassmannians

n-planes in $(n+k)$-dim space

(topologised as a direct limit.)

vector bundle $\quad$ also

$$\begin{tikzcd} f^*EGL \arrow[r] \arrow[d] & EGL(n,\mathbb{R}) \arrow[d] \\ f: X \arrow[r] & BGL(n,\mathbb{R}) \end{tikzcd}$$

1) $X = * \Rightarrow Vect_n(*) = \{ \mathbb{F}^n \times * \longrightarrow * \}$

$\Rightarrow Vect(*) \cong \mathbb{N}$.

$\Rightarrow K(*) \cong \mathbb{Z}$

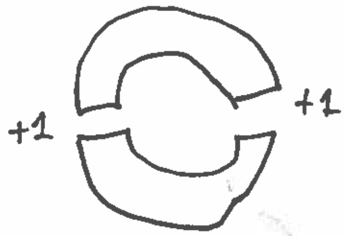
(2) $X = S^1$. Now it matters if we use $\mathbb{R}$ or $\mathbb{F}$.

$S^1 = D^1 \cup_{S^0} D^1$. Any v.b. over D^1 is trivial (up to iso) as $D^1 \simeq *$

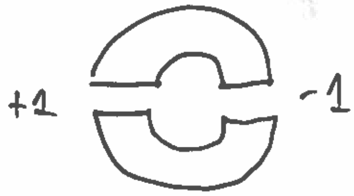
$\underline{\mathbb{F}=\mathbb{R}} \Rightarrow GL(n,\mathbb{R})$ 2 conn components

so when we glue along S^0 the determinants' signs agree, or not.

e.g. $n=1$



$\Rightarrow$ triv



$\Rightarrow$ möbius

So in each dimension there is a trivial and a non-trivial bundle.

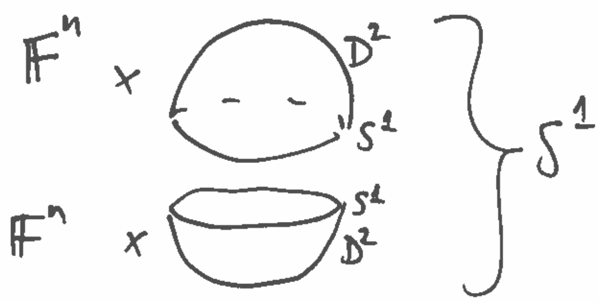
$\Rightarrow K_{\mathbb{R}}(S^1) \cong \mathbb{Z}_2 \oplus \mathbb{Z}$

möbius or not $\quad$ dimension

$\underline{\mathbb{F}=\mathbb{C}}$ $\Rightarrow$ $GL(n,\mathbb{C})$ is conn, so all bundles over S^1 are trivial.

$$K_{\mathbb{C}}(S^1) \cong \underset{\uparrow \text{ dimension}}{\mathbb{Z}}$$

$\underline{n=2}$ $\underline{\mathbb{F}=\mathbb{R}}$ Again we do a "clutching" construction



We glue by $[f: S^1 \longrightarrow GL(n,\mathbb{R})]$

$n \geq 2 \Rightarrow \pi_1 GL(n,\mathbb{R}) \cong \mathbb{Z}_2$. $\therefore K_{\mathbb{R}}(S^2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}$ ← dimension

$\underline{\mathbb{F}=\mathbb{C}}$ $n \geq 2 \Rightarrow \pi_1 GL(n,\mathbb{C}) \cong \mathbb{Z}$ $\therefore K_{\mathbb{C}}(S^2) \cong \mathbb{Z} \oplus \mathbb{Z}$

At this point we will forget about the dimension:

$$i: * \hookrightarrow X \Rightarrow K(X) \longrightarrow K(*) \cong \mathbb{Z}$$ and this splits

$$\Rightarrow K(X) \cong \mathbb{Z} \oplus \tilde{K}(X)$$

<u>Table</u>

	0	1	2	3	4	5	6	7	8
$\tilde{K}_{\mathbb{R}}(S^n)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$ ...
$\tilde{K}_{\mathbb{C}}(S^n)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$ ...

3) Set $BU = \varinjlim_n BU(n)$ Then $\widetilde{KU}(X) = [X, \mathbb{Z} \times BU]$

$BO = \varinjlim_n BO(n)$ $\widetilde{KO}(X) = [X, \mathbb{Z} \times BO]$.

<u>Spectrum</u>

E_n pointed TOP spaces

structure $\alpha: S^1 \wedge E_n \to E_{n+1}$

Smash $X \wedge Y = X \times Y / X \vee Y$

$S^1 \wedge X \cong \Sigma X$ (reduced suspension)

$\Omega X = \{S^1, X\}$ based maps

$[\Sigma X, Y] \cong [X, \Omega Y]$

$\widetilde{KU}$ and $\widetilde{KO}$ are generalised cohomology theories.

This allowed proofs of :

- Bott periodicity (Atiyah–Bott)
- Hopf invariant 1 (Adams)
- Upper bound on # linearly independent vector fields on a sphere (Adams). $\rightsquigarrow$ <u>Index Theory</u>

Around a similar time Serre noticed that

<u>Th'm</u> (1955) R comm, Noetherian ring with 1.

$\Rightarrow \{\text{f.g. proj } R\text{-modules}\} \longleftrightarrow \left\{\begin{array}{l}\text{locally free sheaves of } \mathcal{O}\text{-modules} \\ \text{constant finite rank on Spec } R\end{array}\right\}$

and

<u>Th'm</u> (1962 Swan) X cmpct Hausdorff

$\left\{\begin{array}{l}\text{f.g. proj modules over} \\ C^0(X) = \Omega^0(X) = \{f : X \to \mathbb{R}\}\end{array}\right\} \longleftrightarrow \{\cancel{\text{finite rk}}\ \mathrm{Vect}_{\mathbb{R}}(X)\}$

$\Gamma(X, E) \longleftarrow\!\shortmid (E \to X)$

This bridge, or "duality" between algebra and geometry motivates the study of K-theory from a different perspective. This new "algebraic K-theory" turns out to be one of the deepest ideas in algebraic topology, going beyond the original motivation and providing a powerful algebraic tool for probing the structure of very abstract fields, often linking disparate areas of algebra, topology, geometry and analysis.

Enter Bass

From this point on, our additive category will be

$$\mathcal{P}(R) = \text{f.g. proj modules over a Noetherian ring } R \text{ with unit}$$

We already saw one notion of "higher K-groups" in topological K-theory. We now describe an a priori entirely different notion of higher K-groups.

$$K_0(R) := K(\mathcal{P}(R)) \quad , \text{ the Grothendieck group.}$$

The idea is that K_1 should compare stable ~~aut~~ automorphisms in $\mathcal{P}(R)$.
i.e. measure the differences between equivalences in $K_0(R)$

Note $\quad GL(n, R) \subset GL(n+1, R) \subset \dots$

Set $\quad GL(R) = \operatorname{colim} GL(n, R)$ the infinite general linear group.

Define $\quad K_1(R) = GL(R)^{ab} = GL(R) / [GL(R), GL(R)]$ (Bass '69)

(4) Quote from Hyman Bass on the invention of K_1:

"The idea was that a bundle on a suspension is trivial on each cone [(cf. clutching function)] so glueing on the 'equator' is defined by an automorphism of the trivial bundle, ..., up to homotopy... Since unipotents are connected to the identity (by a straight line, in fact) they belong to the identity component. This indicated that the algebraic definition should at least include the elementary subgroup of $GL_n(R)$."

Indeed,

<u>Whitehead Lemma</u>

$$[GL(R), GL(R)] = E(R) = \left\{ \begin{matrix} \text{elementary matrices} \\ \text{of all dimensions} \end{matrix} \right\}$$

<u>Corr</u>

$$\Rightarrow K_1(R) = GL(R)/E(R).$$

Quillen and higher K-theory

Bass proved that if R is a Dedekind domain and $\mathfrak{p}$ is a ~~prime~~ maximal ideal then there is a localization exact sequence

<u>Th'm</u>

$$K_1(R/\mathfrak{p}) \to K_1(R) \to K_1(R_{\mathfrak{p}})$$

$$\to K_0(R/\mathfrak{p}) \to K_0(R) \to K_0(R_{\mathfrak{p}}).$$

Compare this to the long exact sequence of a fibration of topological spaces:

$$0 \to F \to E \to X \to 0$$

$$\Rightarrow \quad \dots \to \pi_2(X) \to \pi_1(F) \to \pi_1(E) \to \pi_1(X) \to \pi_0(F) \to \pi_0(E) \to \pi_0(X)$$

This, combined with close analogy between K_0, K_1 and the clutching construction / space suspension in topological K-theory led many to conjecture that algebraic K_1, K_0 were just the first 2 of an infinite family of groups that were each invariants of R, it's automorphisms, the automorphisms of its automorphisms ....

Several correct versions of "higher" K-theory were formulated (Swan, Gersten, Karoubi-Villamayor, Volodin). But the most powerful and lasting construction was due to Quillen.

- +-construction
- Q-construction

$\Big\}$ each construct a topological space s.t. $\pi_n =: K_n(R)$ and K_0, K_1 agree with our previous definitions.

We will not delve too deep into these. But we will say something about the Q-construction.

The Q-construction takes an "exact category", a category with a distinguished collection of SES's (+ conditions), constructs an auxilliary category $Q\mathcal{C}$. The htpy groups of the classifying space of this category are then

$$K_i(\mathcal{C}) := \pi_{i+1}(BQ\mathcal{C}) = \pi_i(\Omega BQ\mathcal{C})$$

(recall $B : \mathcal{C} \longrightarrow \underset{\text{nerve}}{N_\bullet \mathcal{C}} \longrightarrow \underset{\text{geometric realization}}{|\mathcal{C}|}$)

Finally ... Waldhausen!

In terms of directly studying manifolds, the most powerful and versatile formulation to date: "Waldhausen's A-theory". The idea is to construct a K-theory in an algebraic setting that tells you something directly about a topological input.

input: Waldhausen category (cofibrations + weak equivalences)

e.g. X a simplicial set and $r : Y \overset{s}{\leftrightarrows} X$ a retract with section s from a simplicial set $Y = X +$ finitely many simplices

$$\Rightarrow {}_{\text{cb}}\mathcal{C} = \{(Y, r, s)\}$$

ouput : $A(X) \simeq \Omega$ (certain bisimplicial set associated to $\mathcal{C}$)

(Can also express this via Quillen's +-construction)

So in some ways we come full circle to the classical algebraic K-theory, with topological spaces as our objects of interest.

Application – Whitehead torsion!

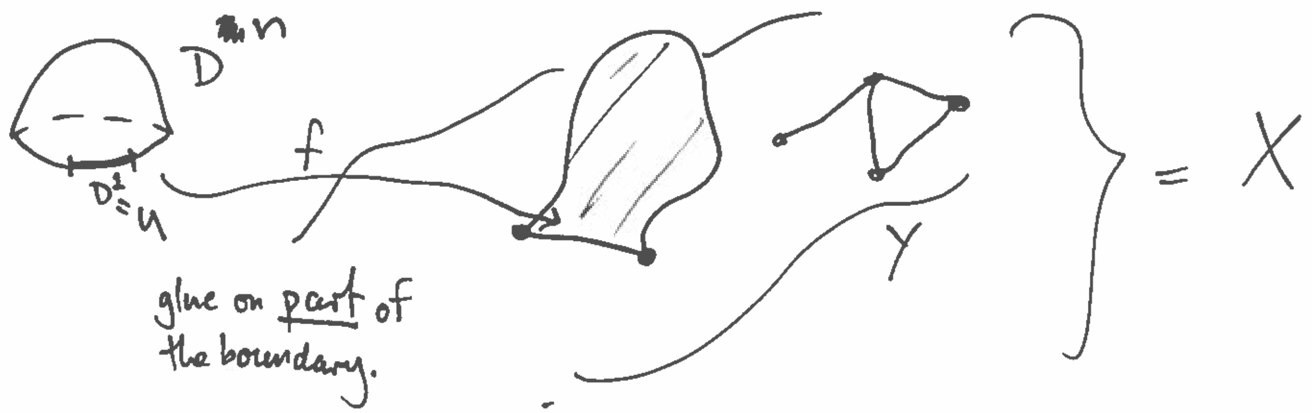
(Actually this example pre-dates the invention of algebraic K-theory.)

Suppose $r: X \xrightarrow{\simeq} Y$ is a deformation retract of CW complexes.

Q: Is X just "Y with a bunch of cells glued on?"

Def'n $Y \nearrow X$ and $X \searrow Y$ means "elementary expansion / elementary contraction"

$$X = Y \cup_f D^n \text{ where set } U = D^{n-1} \subset \partial D^n \text{ and } f: U \to Y^{(n-1)}$$
$$\text{s.t. } \partial U \subset Y^{(n-2)}$$



If

$$X_0 \nearrow X_1 \nearrow X_2 \searrow X_3 \nearrow \dots \nearrow Y, \text{ then } \underset{\text{mixture}}{X \nearrow\!\!\!\searrow Y}$$

If $r: X \to Y$ is htpy equiv to $X \nearrow\!\!\!\searrow Y$ then r is a simple homotopy equiv.

FACT We can homotope attachments so that $\exists$ CW cx Z s.t.

$$X \nearrow Z \searrow Y.$$

5)

Now $r : X \xrightarrow{\simeq} Y \Rightarrow \pi_*(X) = \pi_*(Y)$.

In particular, choosing a fixed lift of a CW approx to universal ~~...~~ covers, $C_*(\tilde{X};\mathbb{Z})$, $C_*(\tilde{Y};\mathbb{Z})$ are finite chain cxs of f.g. $\mathbb{Z}[\pi_1 X] = R$ -modules

and

$$C_*(\tilde{X}) \underset{\simeq}{\xrightarrow{r_*}} C_*(\tilde{Y}) \longrightarrow \underset{\substack{\| \\ \text{cone}(r_*)}}{C_*(\tilde{Y},\tilde{X})} \simeq 0$$

is a htpy cofibration where $\underset{\substack{\| \\ C}}{\text{cone}(r_*)} \simeq 0$

$\Rightarrow \exists$ chain contraction $s : 0 \simeq 1 : C \longrightarrow C$

i.e. $1 = sd + ds$

Then

$$(s+d) = \begin{pmatrix} d & 0 & 0 & \cdots \\ s & d & 0 & \cdots \\ \cdot & s & d & \\ \vdots & \vdots & & \ddots \end{pmatrix} : \bigoplus_{\substack{n \in \mathbb{Z}_{\geq 0} \\ \text{odd}}} C_n \longrightarrow \bigoplus_{\substack{n \in \mathbb{Z}_{\geq 0} \\ \text{even}}} C_n$$

is an isomorphism of f.g. proj $\mathbb{Z}\pi$-modules

i.e. $[s+d] \in GL(\mathbb{Z}\pi)$ (note we had to fix a basis)

<u>Def'n</u> $\tau(X,Y) := [s+d] \in K_1(\mathbb{Z}\pi)$ WHITEHEAD TORSION

<u>Note</u> A <u>lot</u> of choices have been made here.

- Choice of basis of chain modules
- Choice of chain contraction
- Orientations of cell attachment
- base point for universal cover

The point

- Elementary expansions $\longleftrightarrow$ multiplication by elementary matrices $A \in E(R)$.

- Changing orientation of cells or base-point $\longleftrightarrow$ multiplication by

$$\begin{pmatrix} 1 & & & & \\ & \ddots & & & 0 \\ & & 1 & & \\ & 0 & & g & \\ & & & & 1 \ddots \end{pmatrix}$$

$g \in \pi_1(X)$.

All other choices irrellevant.

Hence

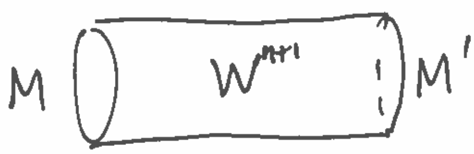
$$\tau(X,Y) \in K_1(\mathbb{Z}\pi) / \{\pm g\} =: \underset{\text{"Whitehead group"}}{Wh}(\pi)$$

well-defined and an obstruction to s.h.e.

Moreover $f: X \to Y$ a homeo, then can show $\tau(X,Y) = 0 \in Wh(\pi)$.

Taking this train of thought and adding the work of Smale from the early 60's you ~~prove~~ can go on to prove s-cobordism th'm.

Th'm $n \geq 5$ CAT = Diff, PL or TOP



s.t. $M \hookrightarrow W$, $M' \hookrightarrow W$ are htpy equivs.

If $\tau(W,M) = 0 \in Wh(\pi)$ then $(W; M, M')$ is a CAT cylinder

In particular, $r: M \xrightarrow{\simeq} M'$ s.t. $\tau(M', M) = 0$ then $M \cong_{CAT} M'$!!!