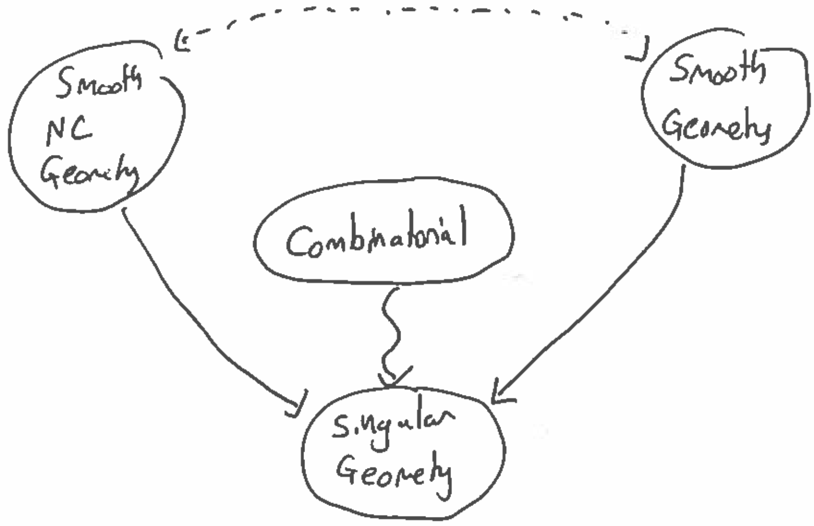
McKay Episode 1: An Overview

Aims: 1) Introduce: 'what is the McKay Correspondence'.
2) Why: 'Why it is interesting'.
3) Example: 'what does it look like!

Intro

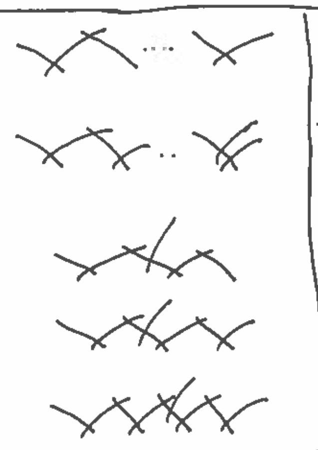
Map:

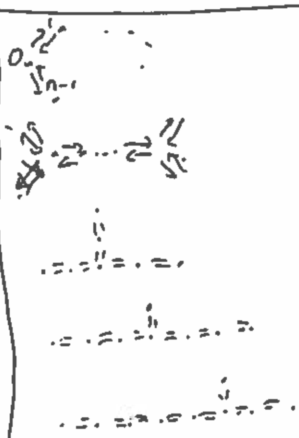


We look at ~~[illegible]~~ $SL_2\mathbb{C}$ case:

Combinatorial object. $G \overset{\text{finite}}{\leq} SL_2\mathbb{C}$

A_{n-1}	Cyclic	$\left\langle \begin{pmatrix} \varepsilon_n & \\ & \varepsilon_n^{-1} \end{pmatrix} \right\rangle$	$XY = Z^n$
D_{n+2}	Binary Dihedral	$\left\langle \begin{pmatrix} \varepsilon_{2n} & \\ & \varepsilon_{2n}^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle$	$X^2+Y^2Z+Z^{n+1}$
E_6	Binary tetrahedral	(see wikipedia)	$X^2+Y^3+Z^4$
E_7	Binary octahedral		$X^2+Y^3+YZ^3$
E_8	Binary icosahedral		$X^2+Y^3+Z^5$





Singular Geometry. $X = \mathbb{C}^2/G = \operatorname{Spec} \mathbb{C}[x,y]^G = \operatorname{Spec} \mathbb{C}[X,Y,Z]/f$

Isolated Affine hypersurface Singularity

$\pi : \tilde{X} \to X$

Smooth Geometry $\tilde{X}$ = crepant resolution
= blow up singular points.

Def.^n $\pi^{-1}(0)$ is the <u>exceptional</u> divisor.
It is a tree of $\mathbb{P}^1$'s.

McKay Quiver

Def^{n} For $(G, \mathbb{C}^2)$ the McKay quiver is defined by:

Vertices: $Q_0 \xleftrightarrow{1:1}$ irreps of G $\quad \{\rho_i\}$

$i \longleftrightarrow \rho_i$

Arrows: $\# \, i \to j = c_{ij}$ $\qquad \rho_i \otimes \mathbb{C}^2 = \bigoplus_{\text{irrep}} \rho_j^{\oplus c_{ij}}$

$[\, c_{ij} = c_{ji} \,]$

Original McKay

The McKay quiver $\setminus$ 0-vertex with $\cdot \rightleftarrows \cdot = \cdot - \cdot$ $=$ Dual graph of Exceptional divisor.

Smooth NC Geometry Think of as some ~~module~~ module category.

Morita equiv. objects
$$\begin{cases} \cdot\ \mathbb{C}[x,y] \rtimes G \\ \cdot\ \hat{A} = \mathbb{C}Q/R \quad \text{preproj alg} \quad (\text{module cat} \simeq \mathrm{Coh}_G\, \mathbb{C}^2) \\ \cdot\ \mathbb{C}[x,y]^G \end{cases}$$

- $\mathbb{C}[x,y] \rtimes G$ $\to$ v.s. $\mathbb{C}[x,y] \otimes \mathbb{C}G$
  $\to$ mult $(f_1 \otimes g_1)(f_2 \otimes g_2) = (f_1 f_2^{g_1} \otimes g_1 g_2)$
  $\rightsquigarrow$ Centre is $\mathbb{C}^2/G$.

- $\hat{A}$ Take path algebra of McKay quiver, label ~~vertex~~ arrows α, α^* s.t. if $\alpha : i \to j$, $\alpha^* : j \to i$
  and impose relations $\sum [\alpha, \alpha^*] = 0$

Example

$\hat{X}$

Affine charts U_0, U_1

$U_0 = \mathrm{Spec}\, \mathbb{C}[x^2, yx^{-1}]$

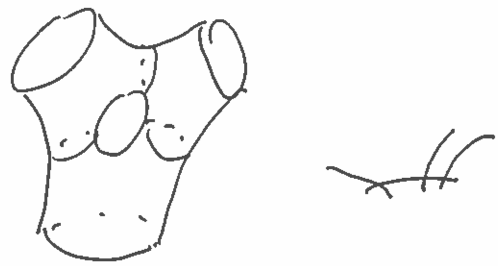
$U_1 = \mathrm{Spec}\, \mathbb{C}[xy^{-1}, y^2]$

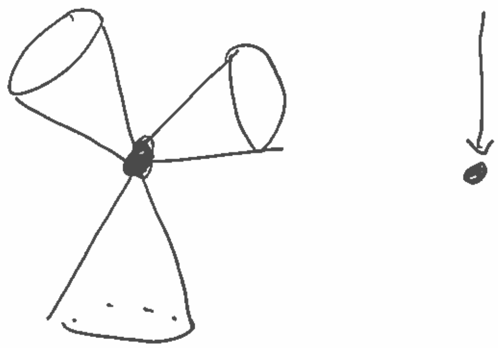
$\mathbb{Z}/2\mathbb{Z} = \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$

$\mathrm{Spec}\, \mathbb{C}[x,y]^{\mathbb{Z}/2\mathbb{Z}}$

$\mathrm{Spec}\, \mathbb{C}[x^2, xy, y^2]$

$\mathrm{Spec}\, \mathbb{C}\dfrac{[X, Y, Z]}{XZ = Y^2}$





Calculations

McKay

$\mathbb{Z}/2\mathbb{Z}$ $\quad \rho_0 \quad \rho_1$

$\mathbb{C}^2 = \rho_1 \oplus \rho_1$

$c_{01} = c_{10} = 2$

$\mathbb{Z}/n\mathbb{Z}$ $\quad \rho_i : g \mapsto \varepsilon_n^i$

$\mathbb{C}^2 = \rho_1 \oplus \rho_{n-1}$

$\rho_i \otimes \mathbb{C}^2 = \rho_{i-1} \oplus \rho_{i+1}$