In this talk I will define G-W invar and quantum product an will discuss some of their properties and examples

Reminder of notations

$\overline{M}_{0,n}(\mathbb{P}^2, d)$ - moduli space of ~~prop~~ stable maps of degree d to $\mathbb{P}^2$ with n marked points.
contracted curves are stable

~~ν_i evaluation map~~

$M_{0,n}(\mathbb{P}^2, d) = \{\mu, p_1, \dots p_n\}$

$\nu_i : M_{0,n}(\mathbb{P}^2, d) \longrightarrow \mathbb{P}^2$
evaluation map $\mu(p_i)$

Consider cohomological classes $h^i \in H^{2i}$
we want: $h^1 \leftrightarrow$ hyperplane, $h^2 = h^1 \cup h^1$ etc $h^2 =$ point $\left(\int_{\mathbb{P}^2} h^2 = 1\right)$
Poincare duality

How many lines go through 2 points? Q_1, Q_2

$M_{0,2}(\mathbb{P}^2, 1)$ $\quad \mathrm{Im}\, \nu_1 = Q_1$
$\mathrm{Im}\, \nu_2 = Q_2$

$\# \nu_1^{-1}(Q_1) \cap \nu_2^{-1}(Q_2)$

~~for general points deg~~

dual picture

$$\# \nu_1^{-1}(Q_1) \cap \nu_2^{-1}(Q_2) \underset{\text{technical}}{=} \int_{M_{0,2}(\mathbb{P}^2, 1)} \nu_1^*(h^2) \cup \nu_2^*(h^2)$$

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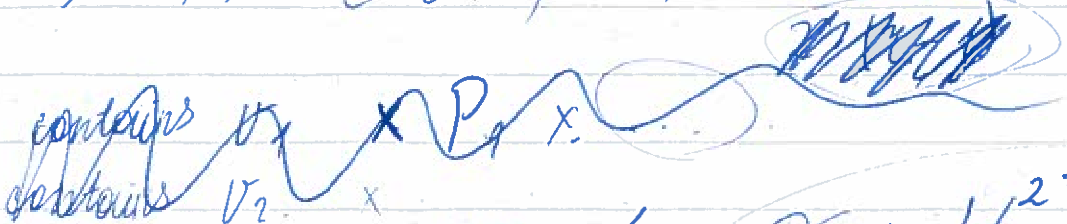
$M_{0,2}(\mathbb{P}^r, 1) = Gr(2, r+1) \times \mathbb{P}^1 \times \mathbb{P}^1$ dim = 2



we get: $\gamma \in H^{2r}$!

can find degree

Definition

$I_d(\gamma_1, \dots, \gamma_n) = \int_{\overline{M}} \cup \, \nu_i^*(\gamma_i)$ — for short $\nu(\gamma)$

1) let c_i be codim of γ_i

$I_d(\gamma_1 \dots \gamma_n)$ makes sense iff $\sum c_i = \dim \overline{M} = rd + r + d + n - 3$

otherwise set $I_d = 0$

By definition I_d is linear so it's enough to calculate $I_d(h^1 \dots h^1, h^2 \dots h^2, \dots h^r \dots h^r)$

$h^i \in A^i = H^{2i}$ $I_d(h^{1 \cdot i_1}, h^{2 \cdot i_2}, \dots h^{r \cdot i_r})$

Proposition if all γ_i have codim ≥ 2

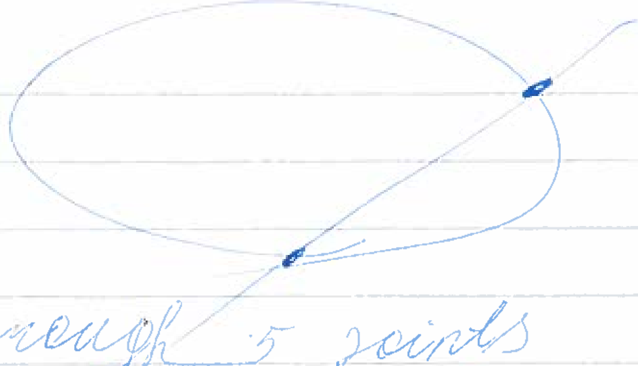
$\Rightarrow I_d(\dots) = \#\{$curves of degree d which meet $\Gamma_i\}$

Γ_i is general collection of subvarieties corresponding to γ_i

2 no proof, based

Example.

$I_d((h^2)^5, h^1) = 2.$



$\exists!$ conic passing through 5 points

2 choices of marking point.

assume $r \geq 2$

Proposition $I_d(\gamma_1, \dots \gamma_n, h) = d\, I(\gamma_1, \dots \gamma_n)$

d points of intersection, d choices of marked points

Lemma ~~for $n \leq 2 \Rightarrow$ nonzero GW I are~~

~~$I_1(h^2, h^2) = 1$~~ for $d = 0$ nonzero

~~$I_1(h)\, d >$~~ GW I are of the form

$I_0(\gamma_1, \gamma_2, \gamma_3) \quad \sum_{i=1}^{3} \operatorname{codim} \gamma_i = 2$

□

$\overline{M} = \overline{M}_{0,n} \times P^2$ $\qquad \gamma_i$ is pr on P^2

(moduli of marked st curves) $\Rightarrow n \geq 3$

$I(\gamma) = \int_{\overline{M}_{0,n} \times P^2} \gamma(\gamma) = \gamma(\gamma) \cdot (\overline{M}_{0,n} \times P^2) =$

$= (\gamma_1 \cup \dots \cup \gamma_n) \cdot pr_*(\overline{M}_{0,n} \times P^2)$

$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \| $
$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad 0$ if $n \geq 4$ (dim.)

■

Lemma for $n \leq 2$ nonzero GWI are

$$I_1(h^n \cdot h^n) = 1$$

□ $d > 0 \Rightarrow \dim \bar M = nd + n + d + n - 3 \geq 2n + n - 2$ ⇒ $n = 2$

and it works for $d = 1$. ■

$$\begin{array}{ccc} \bar M_{0,n+1}(\mathbb{P}^n, d) & \xrightarrow{\ \bar\nu_i\ } & \mathbb{P}^n \\ \downarrow \varepsilon & \nearrow \nu_i & \\ \bar M_{0,n}(\mathbb{P}^n, d) & & \end{array}$$

Lemma nonzero GWI with fund class are

$$I_0(h_0, h^i, h^{n-i})$$

□ $\bar\nu_i^*(h^i) \cap [\bar M_{0,n+1}(\dots)] = \nu(h) \cdot \varepsilon_* [M_{0,n+1}(\dots)] = 0$

We do not have to worry about h^0 and h^1.

On $\mathbb{P}^2$ we only need $I_d(\underbrace{h^2, \dots, h^2}_{3d-1}) = N_d$.

There is recursive algorithm of calculating G-W I

Generally we need existance of invariants and $H^*(V)$ be generated by H^2

(for K_3 surf might not work)

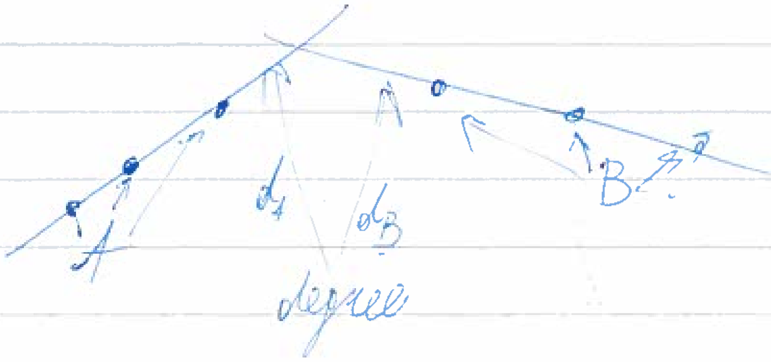
(usually works well for Fano)

It is true for associativity overall.

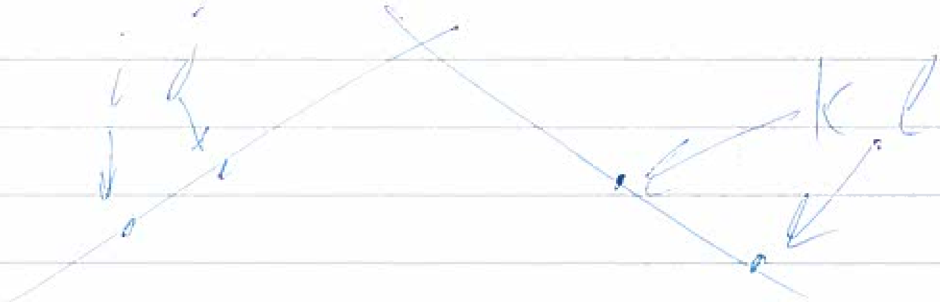
~~I wonder what it means ...~~

~~Proof of Kontsevich formula~~

$D(A, B, d_A, d_B)$



$D(ij|kl)$



$$D(ij|kl) = \sum_{\substack{d_A + d_B = d \\ \{i,j\} \in A \\ \{k,l\} \in B}} D(A, B, d_A, d_B)$$

$$D(ij|kl) \sim D(ik|lj)$$

|| numerically

Lemma $\int_{D(A,B,d_A,d_B)} (\nu_1^{-1}\gamma_1 \cup \dots \cup \nu_n^{-1}\gamma_n) =$

$$= \sum_{e+f=r} I_{d_A}(\dots, h^e)\, I_{d_B}(\dots, h^f)$$

classes corresponding to points from A | -//- from B

Combine all three to get recursive relations.

Let us do it for $\mathbb{P}^2$.

λ_1, λ_2 points corresponding to h^1

μ_1, μ_2 points corresp to some h^2

$3d$ classes overall

~~$\lambda_1, \lambda_2 \in A$~~

$$\int_{D(\lambda_1\lambda_2|\mu_1\mu_2)} \dots = \sum_{\substack{d_A+d_B=d \\ \lambda_1,\lambda_2\in A \\ \mu_1,\mu_2\in B}} \int_{D(A,B,d_A,d_B)} \nu(\gamma) =$$

$$= \sum_{\substack{d_A+d_B=d \\ \lambda_1,\lambda_2\in A \\ \mu_1,\mu_2\in B}} \sum_{e+f=2} I_{d_A}(h^1 h^1 \dots h^e)\, I_{d_B}(h^2, \dots h^2 h^e) =$$

~~how many partitions?~~

$$= N_d + \sum_{\substack{d_A+d_B=d \\ \lambda_1\lambda_2\in A,\ \mu_1\mu_2\in B}} I_{d_A}(h^1 h^1 h^1 \dots) \cdot I_{d_B}(h^1, \dots) =$$

1) get rid of $d_B = 0$ $\quad I_0(h^2 h^2 \dots) = 0$ (dim reason)

2) $d_A = 0 \Rightarrow I_0(h^1 h^1 h^0) \cdot I_d(h^2, \dots h^2)$ where $I_0(h^1h^1h^0) = 1$ and $I_d(h^2,\dots,h^2)$ with $3d-1$ classes $= N_d$

3) from now on $e = f = 1$.

$$I_{d_1}(h^1, h^1, h^1, \underbrace{h^2 \ldots h^2}_{3d_A - 1}) = d_A^3 \cdot N_{d_A}$$

overall we have $3d_A - 4$ ~~since~~ h^2 to put

~~d~~ $\binom{3d-4}{3d_A-1}$

$$= \left(\sum_{\substack{d_A + d_B = d \\ >0 \quad >0}} \binom{3d-4}{3d_A-1} d_A^3 d_B N_{d_A} N_{d_B} \right) + N_d.$$

~~&~~ Similarly $\int_D \ldots$ $\sum_{\substack{d_A + d_B \\ >0 \quad >0}} \binom{3d-4}{3d_A-2} d_A^2 d_B^2 N_{d_A} N_{d_B}$

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Quantum cohomology

$) := \sum_{d=0}^{\infty} I_d(h)$ ($\neq 1$ nonzero)

$) = \Phi(x_0, \ldots, x_n) = \sum_a I((h^0)^{a_0} \ldots (h^n)^{a_n}) \cdot \frac{x_0^{a_0} \ldots x_n^{a_n}}{a_0! \ldots a_n!}$

or in short $\sum I(h^a) \frac{x^a}{a!}$

$\Phi_i = \sum \frac{x^a}{a!} I(h^a, h^i)$ ← extra

P^2 $\Phi(x_0, x_1, x_2)$ у x_0 не нуль [crossed out]

x_1^2 и $x_0 x_0 x_2$ function of x_1 and x_2

$, x_1, x_2) = \frac{x_0 x_1^2}{2} + \frac{x_0^2 x_2}{2} + \left(\sum \quad\right)$

ntum product.

$\ast h^j = \sum_{e+f=2} \Phi_{ij-e} h^f$

~~Associativity of quantum product follows from recursive algorithm of GW I.~~

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~~Let us look at it for $\mathbb{P}^2$~~

~~Prop~~ Lemma h^0 is the identity of quantum product.

Proof for $\mathbb{P}^2$

$$h^1 * h^0 = \sum_{e+f=2} \varphi_{10e} h^f = h^1$$

$e = 1 \Rightarrow f = 1$

similarly for the rest.

~~Let u~~ Th associativity of quantum product is equivalent to ~~...~~ recursive algorithm for G-W I

☐ Let us look at that for $\mathbb{P}^2$:

$$h^1 * h^1 = h^2 + \varphi_{111} h^1 + \varphi_{112} h^0$$

$$h^1 * h^2 = \varphi_{121} h^1 + \varphi_{122} h^0$$

$$h^2 * h^2 = \varphi_{221} h^1 + \varphi_{222} h^0$$

$$(h^1 * h^1) * h^2 = h^1 * (h^1 * h^2) \quad \text{skip calculation}$$

$$\varphi_{222} + \varphi_{111}\varphi_{122} = \varphi_{112} \cdot \varphi_{112}.$$

① ② ③

look at coefficient at X^{3d-4}.

① $\frac{1}{(3d-4)!} I(\underbrace{h_2 \ldots h_2}_{3d-4}, h_1, h_1, h_1) = \frac{N_d}{(3d-4)!}$

② set $d_A + d_B = d$

~~at~~ at X^{3d_A-1} we have. for φ_{111}

$$\frac{1}{(3d_A-1)!} I_{d_A}(\underbrace{h_2, \ldots h_2}_{3d_A-1}, h_1, h_1, h_1) = \frac{d_A^3 \cdot N_{dA}}{(3d_A-1)!}$$

for φ_{122}. ~~at have~~ at X^{3d_B-3} we have.

$$\frac{1}{(3d_B-3)!} I(\underbrace{h_2, \ldots h_2}_{3d_B-3}, h_2, h_1, h_1) = \frac{d_B \cdot N_{d_B}}{(3d_B-3)!}$$

we will have sum over $d_A + d_B = d$

③ believe me we will have.

at X^{3d_A-2} $\frac{d_A^2 \cdot N_{dA}}{(3d_A-2)!}$

at X^{3d_B-2} $\frac{d_B^2 \cdot N_{d_B}}{(3d_B-2)!}$

put it all together and multiply by $(3d-4)!$ to get Kontsevich formula!

$$N_d + \sum \binom{3d-4}{3d_A-1} d_A^3 \cdot N_{d_A} \cdot N_{d_B} \cdot d_B = \sum \binom{3d-4}{3d_A-2} d_A^2 d_B^2 N_{d_A} N_{d_B}$$

Proof ~~of~~ that has to be done geometrically (as Bena has described)