Cyclic Quotient Surface Singularities

Combinatorics

Hirzebruch-Jung Continued Fractions

Let $r, a \in \mathbb{Z}^+$, $(r,a)=1$, $r > a > 0$

$$\frac{r}{a} = \alpha_1 - \cfrac{1}{\alpha_2 - \cfrac{1}{\cdots}} = [\alpha_1, \ldots, \alpha_k]$$

algorithm
- $x_0 = \frac{r}{a}$
- $\alpha_{i+1} = \lceil x_i \rceil$ stop when $\alpha_{j+1} = x_j$
- $x_{i+1} = \frac{1}{\alpha_{i+1} - x_i}$

Examples
- $7/2 = 4 - 1/2 = [4,2]$
- $7/5 = 2 - 3/5 = 2 - \frac{1}{5/3} = 2 - \frac{1}{2 - 1/3} = [2,2,3]$
- $n/1 = [n]$
- $\frac{n}{n-1} = [\overbrace{2, \ldots, 2}^{n-1}]$

Riemenschneider Duality:

$$\frac{r}{a} = [\alpha_1, \ldots, \alpha_k] \qquad \sum (\alpha_i - 1) = \sum (\beta_i - 1)$$

$$\frac{r}{r-a} = [\beta_1, \ldots, \beta_l]$$

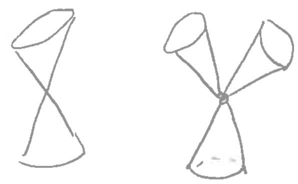
Geometry

Set Up

$G = \frac{1}{r}(1,a) := \left\langle \begin{pmatrix} \varepsilon_r & \\ & \varepsilon_r^a \end{pmatrix} \right\rangle$ ε_r primitive r^{th} root of 1, $(r,a)=1$, $0 < a < r$

- $\frac{r}{a} = [\alpha_1, \ldots, \alpha_k]$
- $\frac{r}{r-a} = [\beta_1, \ldots, \beta_l]$

G acts on $\mathbb{C}^2$ $X = \mathbb{C}^2/G$



Cyclic quotient singularity

Describe X

X is affine $\mathbb{C}[X] = \mathbb{C}[x,y]^G = A$, $X = \operatorname{Spec} A$

(G acts on $\mathbb{C}^2$, $\mathbb{C}[x,y] = \operatorname{Sym}^\bullet(\mathbb{C}^{2\vee})$)

Give generators + rel^{n}s for A

Generators $\frac{r}{r-a} = [\beta_1, \dots, \beta_l]$

$Z_i \quad i = 0,1,\dots,l+1 \qquad Z_0 = x^r \qquad Z_i^{\beta_i} = Z_{i-1}Z_{i+1}$

$Z_1 = x^{r-a}y$

Examples

- $\frac{1}{7}(1,2) \qquad \frac{7}{5} = [2,2,3] \qquad Z_0 = x^7,\ Z_1 = x^5y,\ Z_2 = x^3y^2,\ Z_3 = xy^3,\ Z_4 = y^7$
- $\frac{1}{n}(1,n-1) \qquad \frac{n}{1} = [n] \qquad Z_0 = x^n,\ Z_1 = xy,\ Z_2 = y^n$
- $\frac{1}{7}(1,5) \qquad \frac{7}{2} = [4,2] \qquad Z_0 = x^7,\ Z_1 = x^2y,\ Z_2 = xy^4,\ Z_3 = y^7$
- $\frac{1}{n}(1,1) \qquad \frac{n}{n-1} = [\underbrace{2,\dots,2}_{n-1}] \qquad Z_0 = x^n,\ Z_1 = x^{n-1}y,\ \dots,\ Z_i = x^{n-i}y^i,\ \dots,\ Z_n = y^n$

Relations Among Z_i, generated by 2×2 pseudominors of

$$\begin{pmatrix} Z_0 & & Z_1 & & Z_2 & \cdots & & Z_l \\ & Z_1^{\beta_1-2} & & Z_2^{\beta_2-2} & & \cdots & Z_l^{\beta_l-2} & \\ Z_1 & & Z_2 & & & & Z_l & Z_{l+1} \end{pmatrix}$$

Examples

- $\frac{1}{7}(1,2) \qquad \begin{pmatrix} Z_0 & Z_1 & Z_2 & & Z_3 \\ & & & Z_3 & \\ Z_1 & Z_2 & Z_3 & & Z_4 \end{pmatrix}$
- $\frac{1}{7}(1,5) \qquad \begin{pmatrix} Z_0 & & Z_1 & Z_2 \\ & Z_1^2 & & \\ Z_1 & & Z_2 & Z_3 \end{pmatrix}$
- $\frac{1}{n}(1,n-1) \qquad \begin{pmatrix} Z_0 & & Z_1 \\ & Z_1^{n-2} & \\ Z_1 & & Z_2 \end{pmatrix} \qquad Z_0Z_2 - Z_1^n$ $\boxed{\text{hypersurface}}$
- $\frac{1}{n}(1,1) \qquad \begin{pmatrix} Z_0 & Z_1 & Z_2 & \cdots & Z_l \\ Z_1 & Z_2 & & \cdots & Z_{l+1} \end{pmatrix}$ $\boxed{\text{Determental}}$

Note

- $\frac{1}{2}(1,1)$ $\qquad z_0, z_1, z_2 \qquad z_0 z_2 = z_1^2$

  $\rightsquigarrow$ Nilpotent trace zero matrices $\begin{pmatrix} a & -c \\ b & -a \end{pmatrix}$ $\xrightarrow{\text{deform}}$ $\begin{pmatrix} a & -c \\ b & -d \end{pmatrix}$ Atiyah Flop / conifold

  $a^2 = bc \qquad ad = bc$

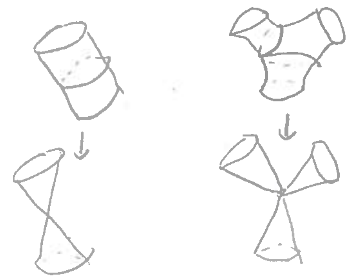
- $\frac{1}{n}(1,1)$ $\qquad H^0(\mathbb{P}^1, \mathcal{O}(n)) = \{x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n\}$

  n^{th} veronese

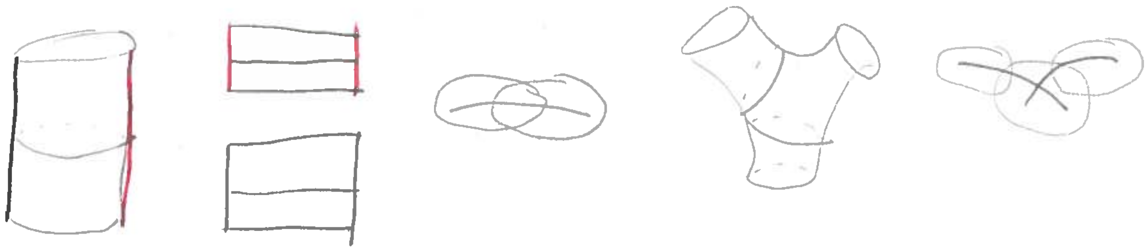
Describe the resolution

$\tilde{X}$
- charts
- map $\tilde{X} \to X$
- exceptional divisors

$\tilde{X} \to X$



Charts



$\frac{1}{r}(1,a) \qquad \frac{r}{a} = [\alpha_1, \ldots, \alpha_k] \qquad U_i = \operatorname{Spec} \mathbb{C}[X_i, Y_i]$

$$X_{i+1} = Y_i^{-1}$$
$$Y_{i+1} = Y_i^{\alpha_{i+1}} X_i$$

map $\qquad X_0 = x^r \qquad Y_0 = y x^{-a}$

Example

- $\frac{1}{7}(1,2)$

  $\frac{7}{2} = [4, 2]$

$U_0 \quad X_0 = x^7 \quad Y_0 = y x^{-2}$

$U_1 \quad X_1 = x^2 y^{-1} \quad Y_1 = y^4 x^{-1}$

$U_2 \quad X_2 = x y^{-4} \quad Y_2 = y^7$

- $\frac{1}{n}(1,1)$ $\quad \frac{n}{1} = [n]$

U_0	x^n	yx^{-1}
U_1	xy^{-1}	y^n

- $\frac{1}{n}(1,n-1)$ $\quad \frac{n}{n-1} = [\overbrace{2,\dots,2}^{n-1}]$

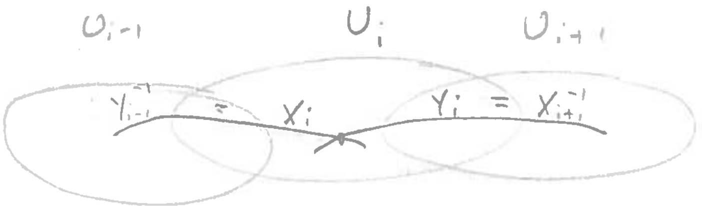
U_0	x^n	$yx^{-(n-1)}$
U_1	$x^{n-1}y^{-1}$	$y^2x^{-(n-2)}$
$\vdots$		
U_i	$x^{n-i}y^{-i}$	$y^{i+1}x^{-(n-i-1)}$
$\vdots$		
U_{n-1}		

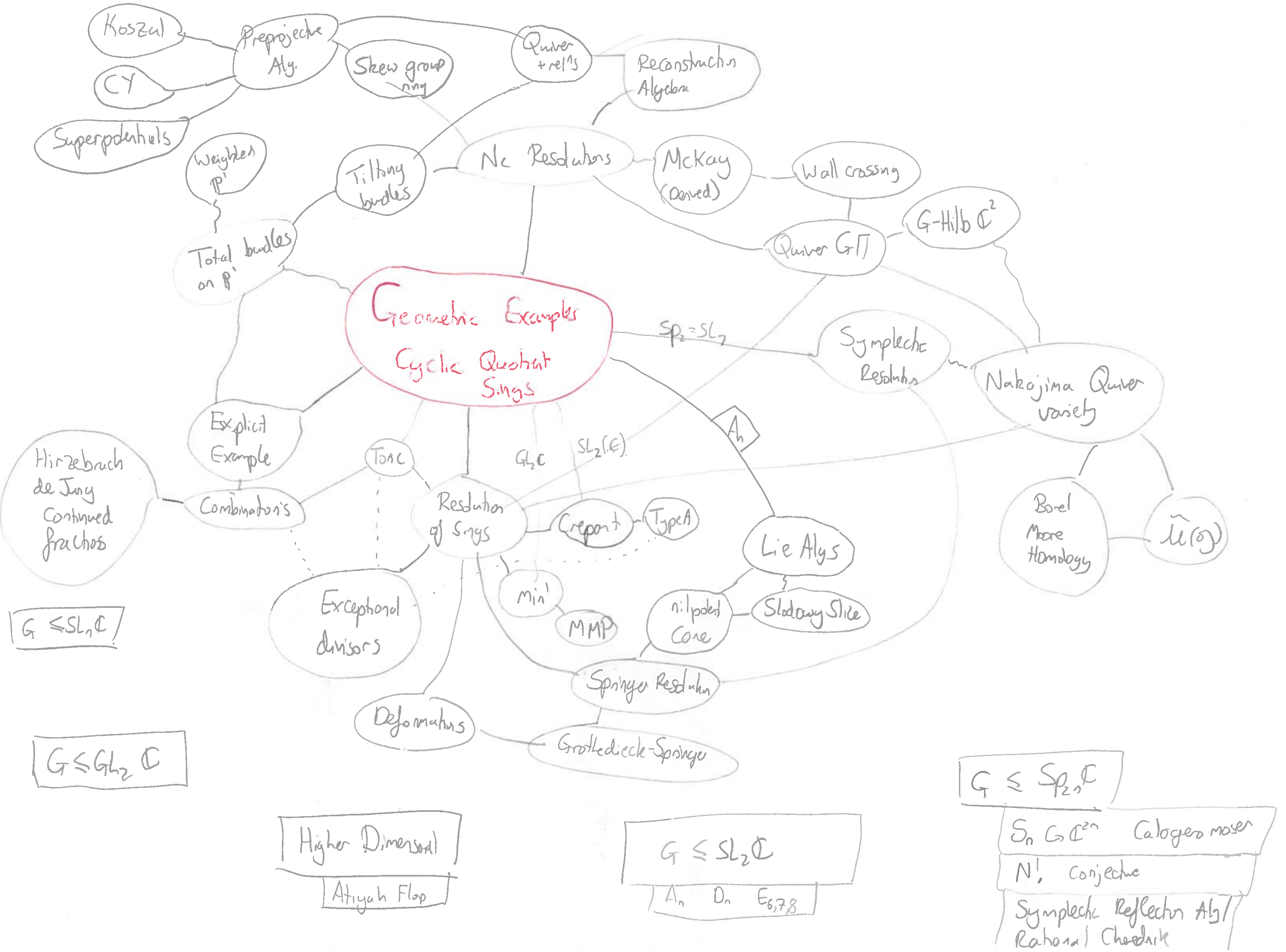
Study map back to X

e.g. $\frac{1}{n}(1,n-1)$ $\quad X \quad x^n, xy, y^n$

	U_0	U_1	$\dots$	U_i	$\dots$	U_{n-1}
x^n	X_0	$X_1^2Y_1$		$X_i^{i+1}Y_i^i$		$X_{n-1}^nY_{n-1}^{n-1}$
xy	X_0Y_0	X_1Y_1		X_iY_i		$X_{n-1}Y_{n-1}$
y^n	$X_0^{n-1}Y_0^n$	$X_1^{n-2}Y_1^{n-1}$		$X_i^{n-i-1}Y_i^{n-i}$		Y_{n-1}

Find exceptional divisors





Geometric Examples
Cyclic Quotient Sings
Koszul
Preprojective Alg.
CY
Superpotentials
Skew group ring
Quiver + rel'ns
Reconstruction Algebra
Weighted $\mathbb{P}^1$
Tilting bundles
NC Resolutions
McKay (Derived)
Wall crossing
Total bundles on $\mathbb{P}^1$
Quiver GIT
G-Hilb $\mathbb{C}^2$
$Sp_2 = SL_2$
Symplectic Resolutions
Nakajima Quiver variety
Explicit Example
Toric
$GL_2\mathbb{C}$
$SL_2(\mathbb{C})$
A_n
Hirzebruch de Jung Continued fractions
Combinatorics
Resolution of Sings
Crepant
Type A
Lie Algs
Borel Moore Homology
$\widetilde{\mathcal{U}}(\mathfrak{g})$
min'l
MMP
nilpotent Cone
Slodowy Slice
Exceptional divisors
Springer Resolution
Deformations
Grothendieck-Springer
$G \leq SL_n\mathbb{C}$
$G \leq GL_2\mathbb{C}$
Higher Dimensional
Atiyah Flop
$G \leq SL_2\mathbb{C}$
A_n D_n $E_{6,7,8}$
$G \leq Sp_{2n}\mathbb{C}$
$S_n \curvearrowright \mathbb{C}^{2n}$ Calogero moser
N! Conjecture
Symplectic Reflection Alg / Rational Cherednik