Model Categories

Aim: Generalise homotopy theory to more abstract contexts.

In homotopy theory, there is a class of maps called "weak equivalences" that we would like to think of as isomorphisms. How can we formalise this?

One approach is the following: Given a category $\mathscr{C}$ and a class of maps $\mathcal{W}$ in $\mathscr{C}$, we define the localisation of $\mathscr{C}$ at $\mathcal{W}$, $\mathscr{C}[\mathcal{W}^{-1}]$ by formally adjoining inverses to all the maps in $\mathcal{W}$.

There are a few ~~two~~ downsides to this approach:

- It is very difficult to describe what the morphisms in $\mathscr{C}[\mathcal{W}^{-1}]$ are in general
- The hom "sets" in $\mathscr{C}[\mathcal{W}^{-1}]$ may be proper classes.
- This does not generalise homotopy theory that well: there should also be a homotopy relation between maps, not just a bunch of weak equivalences.

The purpose of model categories is to identify a context in which we can do this and still have a nice description of the maps in $\mathscr{C}[\mathcal{W}^{-1}]$, and describe them in terms of a homotopy relation.

Definitions:

Given a category $\mathscr{C}$, the arrow category $\mathscr{C}^2$ has as objects the morphisms of $\mathscr{C}$, and a morphism in $\mathscr{C}^2$ $f \to g$ consists of a pair of morphisms (u,v) such that

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "u"'] & \cdot \arrow[d, "v"] \\ \cdot \arrow[r, "g"'] & \cdot \end{tikzcd}$$

commutes.

We say that f is a retract of g if there is a diagram

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "1"] & \bullet \arrow[r] \arrow[d, "g"] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r] \arrow[rr, bend right, "1"'] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

i.e. there is a split epi $g \to f$ in $\mathcal{C}^2$.

A functorial factorization on $\mathcal{C}$ consists of a pair of functors $\alpha, \beta : \mathcal{C}^2 \longrightarrow \mathcal{C}^2$ such that $f = \beta(f) \circ \alpha(f)$ for all $f \in \mathcal{C}^2$

So (α, β) takes a diagram

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "u"'] & \bullet \arrow[d, "v"] \\ \bullet \arrow[r, "g"'] & \bullet \end{tikzcd} \longmapsto \begin{tikzcd} \bullet \arrow[r, "\alpha(f)"'] \arrow[d, "u"'] \arrow[rr, bend left, "f"] & \bullet \arrow[r, "\beta(f)"'] \arrow[d] & \bullet \arrow[d, "v"] \\ \bullet \arrow[r, "\alpha(g)"] \arrow[rr, bend right, "g"'] & \bullet \arrow[r, "\beta(g)"] & \bullet \end{tikzcd}$$

Suppose $i : A \to B$, $p : X \longrightarrow Y$ are maps in $\mathcal{C}$.
We say i has the left lifting property w.r.t. p (p has the rlp w.r.t. i) if for every square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

there is ~~[illegible]~~ some h making both triangles commute.

A model structure on $\mathcal{C}$ consists of three subcategories (i.e. classes of maps containing identities and closed under composition) called weak equivalences, cofibrations and fibrations such that

- If f, g are morphisms in $\mathcal{C}$, and two out of $f, g, g \circ f$ are weak equivalences, so is the third (2 out of 3)
- each class of maps is closed under retracts.
- Trivial cofibrations have llp w.r.t. fibrations
  cofibrations have llp w.r.t. trivial fibrations
  where $\{\text{trivial cofibrations}\} = \{\text{cofibrations}\} \cap \{\text{w.e.}\}$.
  and dually.
- There are functorial factorisations (α, β), (γ, δ) such that
  $\alpha(f)$ is a cofibration, $\beta(f)$ is a trivial fibration
  $\gamma(f)$ is a trivial cofibration, $\delta(f)$ is a fibration
  for any f.

A model category is a complete and cocomplete category together with a model structure.

<u>Example</u> Most examples that are interesting require more machinery to construct. However there is one non-trivial example that is easy to describe.

A cts map $p : X \to Y$ is a Hurewicz fibration if it has the homotopy lifting property (w.r.t. all spaces), i.e. any homotopy between maps pf, pg can be lifted to a homotopy $f \to g$.

$i : A \to B$ is a Hurewicz cofibration if it has the homotopy extension property (defined dually).

Then (homotopy equivalences, Hurewicz cofibs, Hurewicz fibs) is a model structure on Top.

This was proved by Arne Strøm in "the homotopy category is a homotopy category".

The homotopy category of a model category $\mathcal{C}$ is defined to be $\mathrm{Ho}\,\mathcal{C} = \mathcal{C}[W^{-1}]$ where W = weak equivalences.
We do not yet know that $\mathrm{Ho}\,\mathcal{C}$ doesn't suffer from the problems described above. To explain why we must talk about the homotopy relations.

Definitions.
A cylinder object B' for B is an object equipped with a factorisation

$$B \amalg B \xrightarrow{\text{cofib.}} B' \xrightarrow{\text{w.e.}} B$$

of the fold map $B \amalg B \xrightarrow{\nabla} B$. Let i_0, i_1 ~~then~~ denote the maps

$$\begin{tikzcd}
B \arrow[dr, "\iota_0"'] \arrow[drr, bend left, "i_0"] & & \\
& B \amalg B \arrow[r] & B' \\
B \arrow[ur, "\iota_1"] \arrow[urr, bend right, "i_1"'] & &
\end{tikzcd}$$

~~A path object X' consists of~~

A path object X' for X consists of an object with a factorisation

$$X \xrightarrow{\text{w.e.}} X' \xrightarrow{\text{fib.}} X \times X$$

of the diagonal $X \xrightarrow{\Delta} X \times X$. Write

$$\begin{tikzcd}
& & X \\
X' \arrow[r] \arrow[urr, bend left, "p_0"] \arrow[drr, bend right, "p_1"'] & X \times X \arrow[ur, "\pi_0"'] \arrow[dr, "\pi_1"] & \\
& & X
\end{tikzcd}$$

There is a functorial choice of cylinder object $B \times I$ obtained by
↑ not a product! just notation

$$B \amalg B \xrightarrow{\alpha(\nabla)} B \times I \xrightarrow{\beta(\nabla)} B$$

and a functorial choice of path object X^I

$$X \xrightarrow{\gamma(\Delta)} X^I \xrightarrow{\delta(\Delta)} X \times X.$$

Suppose $f, g : B \to X$.
A left homotopy $f \to g$ is a map $H : B' \to X$ such that $Hi_0 = f$, $Hi_1 = g$.
A right homotopy $f \to g$ is a map $K : B \to X'$ such that $p_0 K = f$, $p_1 K = g$.

$$\begin{tikzcd}
B \arrow[dr] \arrow[drr, "i_0"] \arrow[drrr, bend left, "f"] & & & \\
& B \amalg B \arrow[r] & B' \arrow[r, "H"] & X \\
B \arrow[ur] \arrow[urr, "i_1"'] \arrow[urrr, bend right, "g"'] & & &
\end{tikzcd}
\qquad
\begin{tikzcd}
& & & X \\
B \arrow[r, "K"] \arrow[urrr, bend left, "f"] \arrow[drrr, bend right, "g"'] & X' \arrow[r] \arrow[urr, "p_0"] \arrow[drr, "p_1"'] & X \times X \arrow[ur, "\mathrm{pr}_0"'] \arrow[dr, "\mathrm{pr}_1"] & \\
& & & X
\end{tikzcd}$$

$f \overset{l}{\sim} g$ if there is a left homotopy $f \to g$
$f \overset{r}{\sim} g$ if " right "
$f \sim g$ if both.

<u>Def^n</u> B is cofibrant if $0 \xrightarrow{!} B$ is a cofibration, where 0 = initial object.
X is fibrant if $X \to 1$ is a fibration.

Every object has "cofibrant replacement" $0 \xrightarrow{\alpha(!)} B' \xrightarrow{\beta(!)} B$ that is weakly equivalent to it.
Similarly fibrant replacements.

Lemma If B is cofibrant, X fibrant then $\overset{l}{\sim} = \overset{r}{\sim} = \sim$ in $\mathcal{C}(B, X)$, and is an equivalence relation. Also, it is respected by composition with maps $A \to B$ or $X \to Y$ where A is cofibrant and Y is fibrant.

Corollary The quotient category $\mathcal{C}_{cf}/\sim$ exists, where $\mathcal{C}_{cf}$ is the full subcategory of cofibrant and fibrant objects.

$\mathcal{C}_{cf} \to \mathcal{C}_{cf}/\sim$ inverts homotopy equivalences.
Does it invert weak equivalences?
Proposition Yes. In fact the homotopy equivalences in $\mathcal{C}_{cf}$ are precisely the weak equivalences.

Corollary: There is an isomorphism of categories $\mathcal{C}_{cf}/\sim \cong \mathrm{Ho}\,\mathcal{C}_{cf}$.

Since every object of $\mathcal{C}$ is weakly equivalent to an object of $\mathcal{C}_{cf}$, there is an equivalence of categories $\mathrm{Ho}\,\mathcal{C}_{cf} \simeq \mathrm{Ho}\,\mathcal{C}$.
So we have
Theorem $\mathcal{C}_{cf}/\sim \simeq \mathrm{Ho}\,\mathcal{C}$.

This is good because we know what morphisms in $\mathcal{C}_{cf}$ are. In particular, $\mathrm{Ho}\,\mathcal{C}$ is locally small.

Def^n Let α be an ordinal number, regarded as an ordered set. In particular it is a category, so we can consider functors $\alpha \to \mathcal{C}$.
The colimit of such a functor is called a (transfinite) α-composite.
Intuition: a functor $\alpha \to \mathcal{C}$ is an α-sequence of composable maps in $\mathcal{C}$, and the colimit is the composite.
E.g. $\alpha = \omega$:

$$\begin{tikzcd} X_0 \arrow[r] \arrow[drrrr, bend right] & X_1 \arrow[r] \arrow[drrr, bend right] & X_2 \arrow[r] \arrow[drr, bend right] & X_3 \arrow[r] \arrow[dr] & \cdots \\ & & & & X_\omega \end{tikzcd}$$

Given a class of maps I, define I-cell to be the class of transfinite composites of ~~maps in~~ pushouts of maps in I. The maps of I-cell are called relative I-cell complexes.

A pushout

$$\begin{tikzcd} C_0 \arrow[r, "f"] \arrow[d, "i"'] & X_0 \arrow[d] \\ D_0 \arrow[r] & X_1 \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

(e.g. take $i : S^n \to D^n$)

can be thought of "gluing" D_0 to X_0 via f to obtain X_1.
Thus a map in I-cell is a transfinite sequences of gluings.

I-inj is the class of maps with rlp w.r.t. I.
I-cof " " llp w.r.t. I-inj.

A model category $\mathcal{C}$ is cofibrantly generated if there are <u>sets</u> I, J of maps s.t.
domains of I are small relative to I-cell
" J " " J-cell.

fibrations = J-inj
trivial fibrations = I-inj
cofibrations = I-cof
trivial cofibrations = J-cof.

"smallness" is a technical condition regarding transfinite composites.

<u>Theorem</u> (The small object argument)
Suppose domains of I are small relative to I-cell.
Then there is a functorial factorization (γ, δ) s.t. $\gamma(f) \in I\text{-cell} \subseteq I\text{-cof}$
$\delta(f) \in I\text{-inj}$.

This allows us to construct cofibrantly generated model structures.

The recognition theorem: Given a (co)complete $\mathcal{C}$, a subcat W and sets of maps I, J, there is a model structure on $\mathcal{C}$ with W as weak equivalences, I as generating cofibrations, J as generating trivial cofibrations iff

- W has 2 out of 3
- domains of I are small relative to I-cell
- " J " " J-cell
- $J\text{-cell} \subseteq W \cap I\text{-cof}$
- $I\text{-inj} \subseteq W \cap J\text{-inj}$
- Either $W \cap I \cap I\text{-cof} \subseteq J\text{-cof}$ or $W \cap J\text{-inj} \subseteq I\text{-inj}$.

Examples

Let $\mathcal{C}$ = chain complexes of R-modules for a ring R.
W = homology isomorphisms.

Given a module M, let

$$S^n(M)_k = \begin{cases} M & k = n \\ 0 & \text{o/w} \end{cases}$$

$$D^n(M)_k = \begin{cases} M & k = n \text{ or } n-1 \\ 0 & \text{o/w}. \end{cases}$$

I = ~~maps~~ injections $S^{n-1}(M) \longrightarrow D^n(M)$
J = maps $0 \longrightarrow D^n(M)$.

Then I, J are generating (trivial) cofibrations for a model structure with
W = weak equivalences
fibrations = dimensionwise surjections.
cofibrations = more complicated.
(dimensionwise split injections with cofibrant cokernel)

Let $\mathcal{C} = \mathrm{Top}$, W = weak homotopy equivalences.
I' = boundary inclusions $S^{n-1} \to D^n$
J = inclusions $D^n \to D^n \times I$
$x \mapsto (x, 0)$
cofibrations = I'-cof
fibrations = J-inj.
Fibrations are Serre fibrations.

Let $\mathcal{C}$ = SSet. = $[\Delta^{op}, \mathrm{Set}]$
I = canonical inclusions $\partial\Delta[n] \to \Delta[n]$
J = canonical inclusions $\Lambda^r[n] \to \Delta[n]$
cofibrations = I-cof
fibrations = J-inj = Kan fibrations.
$W = \{f \mid |f| \text{ is a weak equivalence in Top}\}$.