

Model Categories

Aim: Generalise homotopy theory to more abstract contexts.

In homotopy theory, there is a class of maps called "weak equivalences" that we would like to think of as isomorphisms. How can we formalise this?

One approach is the following: Given a category \mathcal{C} and a class of maps W in \mathcal{C} , we define the localisation of \mathcal{C} at W , $\mathcal{C}[W^{-1}]$ by formally adjoining inverses to all the maps in W .

There are ~~two~~ ^{several} downsides to this approach:

- It is very difficult to describe what the morphisms in $\mathcal{C}[W^{-1}]$ are in general.
- The hom "sets" in $\mathcal{C}[W^{-1}]$ may be proper classes.
- This does not generalise homotopy theory that well: there should also be a homotopy relation between maps, not just a bunch of weak equivalences.

The purpose of model categories is to identify a context in which we can do this and still have a nice description of the maps in $\mathcal{C}[W^{-1}]$, and describe them in terms of a homotopy relation.

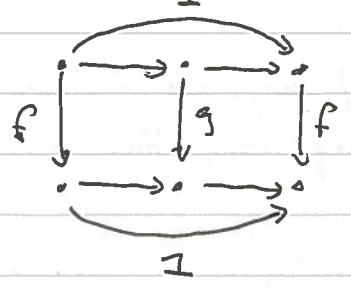
Definitions:

Given a category \mathcal{C} , the arrow category \mathcal{C}^2 has as objects the morphisms of \mathcal{C} , and a morphism in \mathcal{C}^2 $f \rightarrow g$ consists of a pair of morphisms (u, v) such that

$$\begin{array}{ccc} \bullet & \xrightarrow{f} & \bullet \\ u \downarrow & & \downarrow v \\ \bullet & \xrightarrow{g} & \bullet \end{array}$$

commutes.

We say that f is a retract of g if there is a diagram

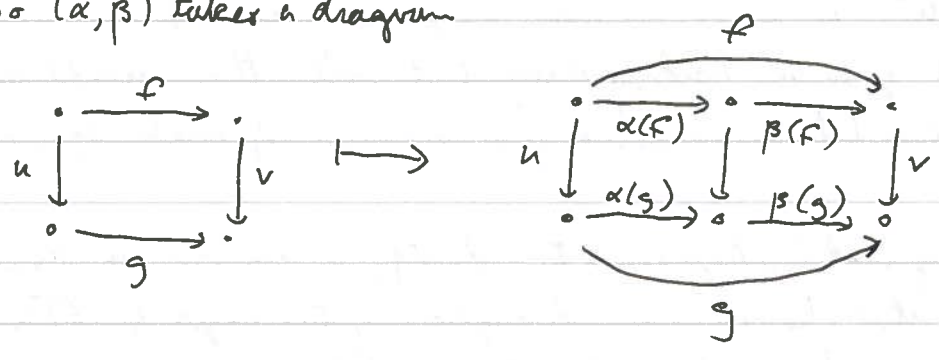


i.e. there is a split epi $g \rightarrow f$ in \mathcal{C}^2 .

A functorial factorisation on \mathcal{C} consists of a pair of functors $\alpha, \beta: \mathcal{C}^2 \rightarrow \mathcal{C}^2$ such that $f = \beta(f) \circ \alpha(f)$ for all $f \in \mathcal{C}^2$

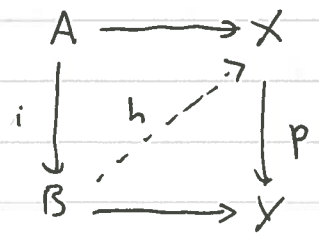
~~Moreover~~

So (α, β) takes a diagram



Suppose $i: A \rightarrow B, p: X \rightarrow Y$ are maps in \mathcal{C} .

We say i has the left lifting property w.r.t. p (p has the rlp w.r.t. i) if for every square



there is ~~some~~ some h making both triangles commute.

A model structure on \mathcal{C} consists of three subcategories (i.e. classes of maps containing identities and closed under composition) called weak equivalences, cofibrations and fibrations such that

- If f, g are morphisms in \mathcal{C} , and two out of $f, g, g \circ f$ are weak equivalences, so is the third (2 out of 3)
- each class of maps is closed under retracts.
- Trivial cofibrations have lfp w.r.t. fibrations
cofibrations have lfp w.r.t. ~~trivial~~ fibrations
where $\{\text{trivial cofibrations}\} = \{\text{cofibrations}\} \cap \{\text{w.e.}\}$
and dually.
- There ~~are~~ ^{are} functorial factorisations $(\alpha, \beta), (\gamma, \delta)$ such that
 $\alpha(f)$ is a cofibration, $\beta(f)$ is a trivial fibration
 $\gamma(f)$ is a trivial cofibration, $\delta(f)$ is a fibration
for any f .

A model category is a complete and cocomplete category together with a model structure.

Example Most examples that are interesting require more machinery to construct. However there is one non-trivial example that is easy to describe.

A map $p: X \rightarrow Y$ is a Hurewicz fibration if it has the homotopy lifting property (w.r.t. all spaces), i.e. any homotopy between maps $p \circ f, p \circ g$ can be lifted to a homotopy $f \rightarrow g$.

$i: A \rightarrow B$ is a Hurewicz cofibration if it has the homotopy extension property (defined dually).

Then (homotopy equivalences, Hurewicz cofibs, Hurewicz fibs) is a model structure on Top .

This was proved by Ane Strøm in "the homotopy category is a homotopy category".

The homotopy category of a model category \mathcal{C} is defined to be $\text{Ho } \mathcal{C} = \mathcal{C}[W^{-1}]$ where $W = \text{weak equivalences}$.

We do not yet know that $\text{Ho } \mathcal{C}$ doesn't suffer from the problems described above. To explain why we must talk about the homotopy relations.

Definitions.

A cylinder object B' for B is an object equipped with a factorisation

$$B \amalg B \xrightarrow{\text{cyl.}} B' \xrightarrow{\text{w.e.}} B$$

of the fold map $B \amalg B \xrightarrow{\nabla} B$. Let i_0, i_1 denote the maps

$$\begin{array}{ccc} B & \xrightarrow{i_0} & B' \\ \downarrow c_0 & \nearrow & \downarrow \\ B \amalg B & \longrightarrow & B' \\ \downarrow c_1 & \searrow & \downarrow \\ B & \xrightarrow{i_1} & B' \end{array}$$

A path object X' consists of

A path object X' for X consists of an object with a factorisation

$$X \xrightarrow{\text{w.e.}} X' \xrightarrow{\text{fib.}} X \times X$$

of the diagonal $X \xrightarrow{\Delta} X \times X$. Write

$$\begin{array}{ccc} & P_0 & \rightarrow X \\ X' & \longrightarrow & X \times X \\ & P_1 & \rightarrow X \end{array}$$

$\nearrow \pi_0$
 $\searrow \pi_1$

There is a functorial choice of cylinder object $B \times I$ obtained by

$$B \amalg B \xrightarrow{\alpha(\nabla)} B \times I \xrightarrow{\beta(\nabla)} B$$

↑ not a product! just notation

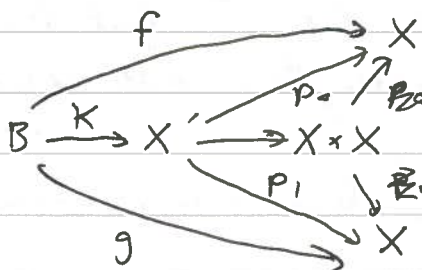
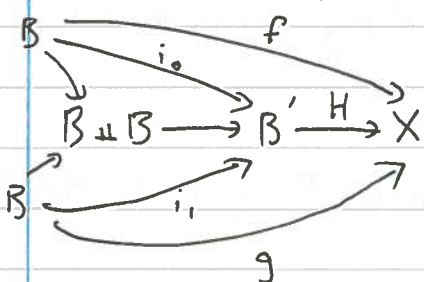
and a functorial choice of path object X^I

$$X \xrightarrow{\gamma(\Delta)} X^I \xrightarrow{\delta(\Delta)} X \times X$$

Suppose $f, g: B \rightarrow X$.

A left homotopy $f \rightarrow g$ is a map $H: B' \rightarrow X$ such that $H i_0 = f, H i_1 = g$.

A right homotopy $f \rightarrow g$ is a map $K: B \rightarrow X'$ such that $p_0 K = f, p_1 K = g$.



$f \overset{l}{\sim} g$ if there is a left homotopy $f \rightarrow g$

$f \overset{r}{\sim} g$ if " right

$f \sim g$ if both.

~~Definition~~

Defⁿ B is cofibrant if $0 \rightarrow B$ is a cofibration, where $0 =$ initial object.
 X is fibrant if $X \rightarrow 1$ is a fibration.

Every object has "cofibrant replacement" $0 \xrightarrow{\alpha(!)} B' \xrightarrow{\beta(!)} B$ that is weakly equivalent to it.

Similarly fibrant replacements.

Given a class of maps I , define I -cell to be the class of transfinite composites of ~~maps~~ pushouts of maps in I . The maps of I -cell are called relative I -cell complexes.

A pushout

$$\begin{array}{ccc} C_0 & \xrightarrow{F} & X_0 \\ i \downarrow & & \downarrow \\ D_0 & \xrightarrow{\quad} & X_1 \end{array} \quad (\text{e.g. take } i: S^n \rightarrow D^n)$$

can be thought of "gluing" D_0 to X_0 via F to obtain X_1 . Thus a map in I -cell is a transfinite sequence of gluings.

~~def~~

I -inj is the class of maps with rlp w.r.t. I .

I -cof " " llp w.r.t. I -inj.

A model category \mathcal{C} is cofibrantly generated if there are sets I, J of maps s.t.

domains of \overline{I} are small relative to I -cell

" " " " " J -cell.

fibrations = J -inj

trivial fibrations = I -inj

cofibrations = I -cof

trivial cofibrations = J -cof.

"smallness" is a technical condition regarding transfinite composites.

Theorem (The small object argument)

Suppose domains of I are small relative to I -cell

Then there is a functorial factorization $\mathcal{C} \xrightarrow{\gamma} \mathcal{C} \xrightarrow{\delta} \mathcal{C}$ s.t. $\gamma(f) \in I\text{-cell} \subseteq I\text{-cof}$
 $\delta(f) \in I\text{-inj}$.

This allows us to construct cofibrantly generated model structures.

The recognition theorem: Given a (co) complete \mathcal{C} , a subclass W and sets of maps I, J , there is a model structure on \mathcal{C} with W as weak equivalences, I as generating cofibrations, J as generating trivial cofibrations iff

- W has 2 out of 3
- domains of I are small relative to I -cell
- " " " " J -cell
- J -cell $\subseteq W \cap I$ -cof
- I -inj $\subseteq W \cap J$ -inj
- Either $W \cap I \cap I$ -cof $\subseteq J$ -cof or $W \cap J$ -inj $\subseteq I$ -inj.

Examples

Let \mathcal{C} = chain complexes of R -modules for a ring R .

W = homology isomorphisms.

Given a module M , let ~~$D^n(M)$~~

$$S^n(M)_k = \begin{cases} M & k=n \\ 0 & \text{o/w} \end{cases}$$

$$D^n(M)_k = \begin{cases} M & k=n \text{ or } n-1 \\ 0 & \text{o/w} \end{cases}$$

I = ^{injections} ~~maps~~ $S^{n-1}(M) \rightarrow D^{n-1}(M)$ ~~$S^{n-1}(M) \rightarrow D^n(M)$~~

J = maps $0 \rightarrow D^n(M)$.

Then I, J are generating (trivial) cofibrations for a model structure with

W = weak equivalences

cofibrations = dimensionwise surjections

cofibrations = more complicated

(dimensionwise split injections with cofibrant ~~of~~ kernel)

Let $\mathcal{C} = \text{Top}$, $W = \text{weak homotopy equivalences}$.

$I' = \text{boundary inclusions } S^{n-1} \rightarrow D^n$

$J = \text{inclusions } D^n \rightarrow D^n \times I$
 $x \mapsto (x, 0)$

cofibrations = I' -inj

fibrations = J -inj.

Fibrations are Serre fibrations.

Let $\mathcal{C} = \text{SSet} = [\Delta^{\text{op}}, \text{Set}]$

$I = \text{canonical inclusions } \partial\Delta[n] \rightarrow \Delta[n]$

$J = \text{canonical inclusions } \Lambda^r[n] \rightarrow \Delta[n]$

cofibrations = I -inj

fibrations = J -inj = Kan fibrations.

$W = \{F \mid |f| \text{ is a weak equivalence in Top}\}$.

