Intersection homology, Part 2

Reminder

~~Last~~ Last time we defined a class~~ed~~ of topological spaces. A stratification of X is a seq of closed subspaces

$$X = X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots \supseteq X_n \supseteq \emptyset$$

- A stratum $S_k := X_k \setminus X_{k+1}$ is a manifold of dimension $n-k$, or empty
- $S_0 = X \setminus X_1$ is dense in X
- We have local normal triviality

  $\forall x \in S_k$ $\exists$ a distinguished neighbourhood $U \ni x$ and a compact stratified space L such that

$$\begin{tikzcd}
U \arrow[r, "\sim"] & \mathbb{R}^{n-k} \times \operatorname{cone}^\circ(L) \\
U \cap X_i \arrow[r, "\sim"] \arrow[u] & \mathbb{R}^{n-k} \times \operatorname{cone}^\circ(L_i) \arrow[u] \\
U \cap X_k \arrow[r, "\sim"] \arrow[u] & \mathbb{R}^{n-k} \times \operatorname{cone}(L_k) \arrow[u]
\end{tikzcd}$$

X is called a pseudomanifold of dim n if there is a stratification with $X_1 = X_2$

We think ~~abc~~ of ~~the~~ $X_1 = X_2 =: \Sigma$ as the singular locus of X and X_3 as the singular locus of $X_2 = \Sigma$ etc.

These are the spaces we would like a sensible homology "theory" for.

<u>Piecewise linear theory</u>

To start we defined the piecewise linear or simplical version of the theory by the chain groups:

$$IC_i^{\bar p}(X) = \left\{ \xi \in C_i^{\bar p}(X) \;\middle|\; \begin{array}{l} \dim |\xi| \cap X_k \leq i-k+p(k) \text{ and} \\ \dim |\partial \xi| \cap X_{k+1} \leq i-k-1+p(k) \end{array} \right\}$$

for simplicial chains with <u>closed</u> support (Borel-Moore)

Q. Why can't we just define singular intersection homology as usual?

A. We can! but it becomes a pain to prove anything including topological invariance.

Sheafification!

We have obvious vector spaces $IC_i^{\bar{p}}(U)$ for each $U \subseteq X$ so the question is, ~~do~~ can we make a sheaf.

Suppose we have $V \subset U$, to define restriction, for a chain $\xi \in C_i(X)$:

- choose a triangulation T of U s.t. ξ is simplicial and a tri. S of V such that every simplex $\sigma' \in S$ is contained in some simplex $\sigma' \in T$.

if $\xi = \sum_{\sigma \in T} \xi_\sigma \sigma$ then define

$$res_V^U \xi := \sum_{\sigma' \in S} \xi^r_{\sigma'} \sigma'$$ by

$$\xi^r_{\sigma'} = \begin{cases} 0 & \text{if } \sigma' \text{ is not contained in an } i\text{-simplex } T \\ \xi_\sigma & \text{if } \sigma' \text{ is contained in } \sigma \in T \end{cases}$$

This defines a map

$$res : C_i(U) \longrightarrow C_i(V)$$

which descends to a map

$$res : IC_i^{\bar{p}}(U) \longrightarrow IC_i^{\bar{p}}(V)$$

checking the other sheaf axioms, we define the IC-sheaf and the chain-sheaf

$$\mathbf{IC}^{-i}_{\bar{p}} : U \mapsto IC_i^{\bar{p}}(U), \quad \mathbf{C}^{-i}_{\bar{p}} : U \longmapsto C_i^{\bar{p}}(U)$$

Note that we have a cochain complex of sheaves:

$$\to IC_{\bar p}^{-i-1} \xrightarrow{\partial} IC_{\bar p}^{-i} \xrightarrow{\partial} IC_{\bar p}^{-i+1} \to$$

We want to take global sections

$$\to IC^{i+1}_{\bar p}(X) \xrightarrow{\partial} IC^{\bar p}_{i}(X) \xrightarrow{\partial} IC^{\bar p}_{i-1}(X) \to$$

and then take homology to get $IH^{\bar p}_i(X)$

summarising: $IH^{\bar p}_i(X) = \cancel{IH^{\bar p}_i}\, H^{-i}(\Gamma(IC^\bullet_i))$

~~To formulate this in the~~ We want to formulate this in the language of derived categories, ie we want to express $IH^{\bar p}_\bullet(X)$ as the image of some derived functor.

Lemma $IC^\bullet_{\bar p}$ and $\mathbb{D}^\bullet_{\bar p}$ are soft complexes of sheaves.

Corollary $\cancel{IH^k_{\bar p}}$ $IH^{\bar p}_i(X) \cong \mathbb{H}^{-i}(X; IC^\bullet_{\bar p})$

we define $IH^i_{\bar p}(X) := \mathbb{H}^i(X; IC^\bullet_{\bar p}) = IH^{\bar p}_{-i}(X)$.

Three
~~TWO~~ IMPORTANT Properties:

Local intersection cohomology.

Prop Let $x \in S_k = X_k \setminus X_{k+1}$ with link L
then the stalk cohomology is given by

i	$-n$	$-n+1$	$\cdots$	$-n+p(k)$
$\mathcal{H}^i_{x}(\mathcal{I}C^\bullet_p)_x$	$IH^p_{k-1}(L)$	$IH^p_{k-2}(L)$	$\cdots$	$IH^p_{k-p(k)-1}(L)$

and zero elsewhere

proof comes from the fact that the stalk cohomology is isomorphic to the cohomology ~~at~~ of the distinguished neighbourhood

then calculate $IH^p_\bullet(\mathbb{R}^{n-k} \times cone^\circ(L))$.

In particular:

$$\mathcal{H}^j(IC^\bullet_p|_{U_{k+1}}) = 0 \quad \text{for } j > p(k) - n.$$

(Vanishing condition)

Reconstructing the IC sheaf

Recall that ~~is~~ for a morphism $f: X \longrightarrow Y$
f_* and f^* are an adjoint pair so there
is always a canonical morphism

$$A \longrightarrow f_* f^* A$$

~~dure~~ define maps

$$i_k : U_k \hookrightarrow U_{k+1} \qquad (U_k = X \setminus X_k)$$

$$j_k : \underset{\overset{\shortparallel}{S_k}}{U_{k+1} \setminus U_k} \hookrightarrow U_{k+1}$$

then

(Attaching property)

<u>Prop</u> the map

$$IC^\circ_p|_{U_{k+1}} \longrightarrow \tau \, i_{k*} i_k^* (IC^\circ_p|_{U_{k+1}})$$

induces isomorphisms

$$H^j(j_k^* IC^\circ_p|_{U_{k+1}}) \xrightarrow{\sim} H^j(j_k^* i_{k*} i_k^* IC^\circ_p|_{U_{k+1}})$$

for $j \leq p(k) - n$

(Normalisation property)

<u>Prop</u> $IC^\circ_p|_{U_2} \cong \mathbb{O}_{U_2}$ (where $\mathbb{O}_M(U) = H^n_c(M, \mathbb{C})$)

(ie for an oriented manifold, $IC^\circ_p|_{U_2} \cong \underline{\mathbb{C}}_{U_2}$)

~~(Constructibility~~

Lets axiomatise these properties!

A sheaf satisfies the IC-axioms if, A

<u>ax I</u> $A^\bullet|_{U_2} \cong \mathbb{Q}_{U_2}$

<u>ax II</u> $H^i(A^\bullet) = 0$ for $i < -n$

<u>ax III</u> $H^i(A^\bullet|_{U_{k+1}}) = 0$ for $i > p(k) - n$

<u>ax IV</u> the maps

$$H^i(j_k^* A^\bullet|_{U_{k+1}}) \longrightarrow H^i(j_k^* R i_{k*} i_k^* A|_{U_{k+1}})$$

are isomorphisms for $i \leq -n + p(k)$

(We can also write the attaching condition as

$$H^i(\mathrm{costalk}_x(A^\bullet)) = 0 \qquad i \leq p(k) - k + 1 \quad x \in S_k$$

where Costalk $= p_x^!$ where $p_x : \{x\} \hookrightarrow X$)

Deligne's Sheaf

We can try and mimic the properties above for a general pseudomanifold X by inductively building a sheaf from a local system.

Def Deligne's sheaf $\mathcal{P}^\circ_{\bar p}(\mathcal{L})$ is defined by

$$\mathcal{P}^\circ = \tau_{\leq p(n)-n} R i_{n*} \tau_{\leq p(n-1)-n} R i_{n-1*} - \tau_{\leq p(2)-n} i_{2*} \mathcal{L}$$

Theorem If A° satisfies the IC-conditions then $A^\circ \cong \mathcal{P}^\circ_{\bar p}(\mathbb{Q}_{U_2})$

Def Now we just take as our definition of intersection homology

$$IH^{\bar p}_i(X) := \mathbb{H}^{-i}(X, \mathcal{P}^\circ_{\bar p}(\mathbb{Q}_{U_2})).$$

This also leads to a proof of topological invariance