Geometry Club 24/1/14 - Intersection Homology Part 1

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Outline:

- Reminder about intersection of submanifolds, Poincaré duality, PL manifolds
- Stratified pseudo-manifolds

History: 1895 - Poincaré defined intersection of chains of comp. dim in a compact manifold.

1925 - Lefschetz - same for chains not of comp. dim.

If x is an i-cycle in a manifold, y a j-cycle, x, y in general position, $x \cap y$ is an $(n-i-j)$-cycle.

This forms a pairing on X manifold

$$Q: C_i(X) \times C_j(X) \to C_{n-i-j}(X)$$

$i+j=n$ then $Q: H_i(X) \times H_j(X) \to H_0(X) \xrightarrow{\varepsilon_*} \mathbb{Z}$

is non-singular over the rationals in the sense that

$$H_i(X) \otimes \mathbb{Q} \xrightarrow{\cong} (H_j(X) \otimes \mathbb{Q})^*.$$

Ex: $M \sim M'$ iff they are bdy of manifold 1 dim bigger

If M is simply connected, $(4k)$-dim then Poincaré duality (intersection pairing) classifies M up to bordism!

Def: A PL structure on a top. manifold is a distinguished set of triangulations $\mathcal{T}$ for X. $T \in \mathcal{T}$ simplicial complex, $|T| \cong X$, $\mathcal{T}$ is closed under barycentric subdivision.

Choose $T \in \mathcal{T}$.

Def: $C_k^T(X) = \{$finite lin. comb. of k simplices in $T\}$.

$\operatorname{colim}_{T \in \mathcal{T}} C_k^T(X) =: C_k(X)$ PL chain group. $\leftarrow H_k(X)$

Def: Borel-Moore chain group of T is

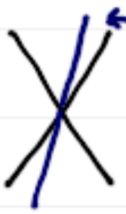
$C_k^T(X) := \{$formal ∞ lin comb. of k simplices$\}$. $\leftarrow H_k^{lf}(X)$

$\cap : C_i^T(X) \times C_j^T(X) \to C_{n-i-j}(X)$.

Ex:  $\mathbb{R}$

i	$H_i(X)$	$H_i^{lf}(X)$
0	$\mathbb{R}$	0
1	0	$\mathbb{R}$
2	0	$\mathbb{R} \oplus \mathbb{R}$

$\sim H_*(Y)$ 



gen of $H_1^{lf}(X)$
NOT in gen. position.

Poincaré duality doesn't work!

If we insist $\dim|\xi \cap \Sigma| \le \dim|\xi| - 2$, fixes problem.

Def: A top. stratified space of dim. n is

- $n = 0$ countable set of pts.
- $n > 0$ X is a Hausdorff paracompact top. space s.t. $\exists$ filtration
  $X = X_n \supset X_{n-1} \supset X_{n-2} \supset X_{n-3} \supset \cdots \supset X_1 \supset X_0 \supset X_{-1} = \emptyset$ s.t
  - k dim stratum $X_k \setminus X_{k-1}$ is a top. manifold of dim @ most k
  - $X \setminus X_{n-1}$ is dense in X.
  - locally normally trivial i.e $\forall x \in X_k \setminus X_{k-1}$ $\exists$ $x \in U \subset X$ and another strat. top space L of dim k s.t $\exists$ homeomorphism

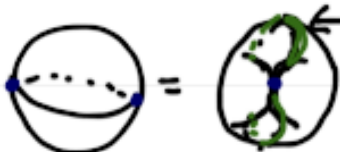
$\phi: \mathcal{U} \xrightarrow{\cong} \mathbb{R}^{n-k-1} \times C(L)$ where $C(L)$ is the open cone on L and ϕ respects the stratifications.

Ex: $\{x=0\} \cup \{y=0\} \subset \mathbb{R}^2$

$X_1 = X$

$X_0 = pt.$

$\mathcal{U}$ is open cone on 4 pts

Ex: 

blue pts ident.

$X_1 = $ loop 

$X_0 = $ pt (blue pt)

Locally  open cone on 4 pts w/ $\mathbb{R}^1$ dir.

Def: A top. strat. pseudo-manifold is a top. strat. space s.t $X_{n-1} = X_{n-2}$.

(means sings. in at least codim 2)

Plan:
- Define $IH_\cdot(X)$ for X a PL pseudomanifold
- Axiomatize some properties
- Extend def. to all top. pseudomanifolds.

Def: A PL pseudomanifold is a PL space $(X, \mathcal{T})$ which is homeomorphic to $\bigcup_{\sigma \in T^n} \sigma$ for some $T \in \mathcal{T}$ such that in T, every $(n-1)$-simplex is the face of exactly 2 n-simplices.

Def: A perversity is a map $p: \{2, \dots, n\} \to \mathbb{Z}$ s.t ① $p(2) = 0$ and ② $p(i) = p(i-1)$ or $p(i-1)+1$.

Ex: $\underline{0}(i)=0$, $\underline{t}(i)=i-2$. $\underline{m}(i)=\lfloor \frac{i}{2}\rfloor - 1$, $\underline{n}(i)=\lceil \frac{i}{2}\rceil - 1$.

$$C_i((X)) = \varinjlim_{T\in\mathcal{T}} C_i^T((X))$$

$$IC_i^{\bar{p}}(X) := \{\xi \in C_i((X)) \mid \dim|\xi|\cap X_{n-k} \le i-k+\bar{p}(k) \text{ and } \dim|\partial\xi|\cap X_{n-k} \le i-k+\bar{p}(k)-1\}.$$

Then $IC_i^{\bar{p}}(X) \xleftarrow{\partial} IC_{i+1}^{\bar{p}}(X)$ is a chain complex.

<u>Def</u>: $IH_i^{\bar{p}}(X) = H_i(IC_\cdot^{\bar{p}}(X))$.

<u>Ex</u>: $X =$ 

$X = X_2 = X_1$

$X_0 =$ pt (blue)

only 1 pt

$C_0(X) = \mathbb{C}^7 \supseteq IC_0(X)$, $\dim \xi\cap X_0 \le -2$

$IC_0(X) = \mathbb{C}^6$ (get rid of sing pt)

$C_1(X) = \mathbb{C}^{12} \supseteq IC_1(X)$, $\dim\xi\cap X_0 \le -1$

$IC_1(X) = \mathbb{C}^6$

$C_2(X) = \mathbb{C}^8 \supseteq IC_2(X)$, $\dim\xi\cap X_0 \le 0$, $\dim\partial\xi\cap X_0 \le -1$.

$IC_2(X) = \mathbb{C}^4$

Get $\mathbb{C}^4 \to \mathbb{C}^6 \to \mathbb{C}^6 \to 0$

ker.	2	2	6
I Hom	2	0	2