The non-multiplicativity of the signature of fibre bundles

Carmen Rovi
www.maths.ed.ac.uk/∼s1142600

University of Edinburgh

Geometry Club
22 November 2013
(Open Day)

Outline

Outline

Outline

1. Statement of the problem
2. Previous results
3. Methods used
4. New result
5. Relations to other ideas

Outline

Outline

Outline

The signature

Definition

The signature of a closed oriented n-dimensional manifold M^n is denoted by $\sigma(M) \in \mathbb{Z}$.

- If $n = 4k$ then $\sigma(M)$ is defined to be the number of positive eigenvalues minus the number of negative eigenvalues of the non-singular symmetric intersection form $(H^{2k}(M;\mathbb{R}), \phi)$, where

$$\phi : H^{2k}(M;\mathbb{R}) \times H^{2k}(M;\mathbb{R}) \longrightarrow \mathbb{R}; (u, v) \mapsto \langle u \cup v, [M] \rangle .$$

- If $n \neq 4k$ then $\sigma(M) = 0 \in \mathbb{Z}$.

Statement of the problem and methods

Let $F^n \longrightarrow E^{n+m} \longrightarrow B^m$ be a fibre bundle of closed, compatibly oriented manifolds with $n + m \equiv 0 \pmod 4$.

The problem

What is the relation between the signatures

$$\sigma(E), \sigma(F), \sigma(B) \in \mathbb{Z} \ ?$$

Methods used in the last 60 years

- The Hirzebruch signature theorem
- Spectral sequences
- Atiyah-Singer index theory
- Characteristic classes
- Group cohomology (cocycles)
- Algebraic K-theory
- Algebraic L-theory

Statement of the problem and methods

Let $F^n \longrightarrow E^{n+m} \longrightarrow B^m$ be a fibre bundle of closed, compatibly oriented manifolds with $n+m \equiv 0 \pmod 4$.

The problem

What is the relation between the signatures

$$\sigma(E), \sigma(F), \sigma(B) \in \mathbb{Z} \ ?$$

Methods used in the last 60 years

- The Hirzebruch signature theorem
- Spectral sequences
- Atiyah-Singer index theory
- Characteristic classes
- Group cohomology (cocycles)
- Algebraic K-theory
- Algebraic L-theory

Some cases where the signature is multiplicative:

For a trivial fibre bundle the signature is multiplicative:

$$\sigma(F \times B) = \sigma(F)\sigma(B) \in \mathbb{Z}$$

Theorem (Hirzebruch, 1953)

If $F^0 \to B^{4k} \to E^{4k}$ is a finite covering of degree $|F^0| = d$ then

$$\sigma(E) = d\sigma(B) \in \mathbb{Z} \quad \textit{(using the signature theorem)}$$

Theorem (Chern, Hirzebruch and Serre, 1957)

If $F^n \to E^{n+m} \to B^m$ is a fibre bundle such that the fundamental group $\pi_1(B)$ acts trivially on $H^*(F; \mathbb{Q})$ then

$$\sigma(E) = \sigma(F)\sigma(B) \in \mathbb{Z} \quad \textit{(using spectral sequences)}$$

Some cases where the signature is multiplicative:

For a trivial fibre bundle the signature is multiplicative:

$$\sigma(F \times B) = \sigma(F)\sigma(B) \in \mathbb{Z}$$

Theorem (Hirzebruch, 1953)

If $F^0 \to B^{4k} \to E^{4k}$ is a finite covering of degree $|F^0| = d$ then

$$\sigma(E) = d\sigma(B) \in \mathbb{Z} \quad \textit{(using the signature theorem)}$$

Theorem (Chern, Hirzebruch and Serre, 1957)

If $F^n \to E^{n+m} \to B^m$ is a fibre bundle such that the fundamental group $\pi_1(B)$ acts trivially on $H^*(F; \mathbb{Q})$ then

$$\sigma(E) = \sigma(F)\sigma(B) \in \mathbb{Z} \quad \textit{(using spectral sequences)}$$

Some cases where the signature is multiplicative:

For a trivial fibre bundle the signature is multiplicative:

$$\sigma(F \times B) = \sigma(F)\sigma(B) \in \mathbb{Z}$$

Theorem (Hirzebruch, 1953)

If $F^0 \to B^{4k} \to E^{4k}$ is a finite covering of degree $|F^0| = d$ then

$$\sigma(E) = d\sigma(B) \in \mathbb{Z}$$ *(using the signature theorem)*

Theorem (Chern, Hirzebruch and Serre, 1957)

If $F^n \to E^{n+m} \to B^m$ is a fibre bundle such that the fundamental group $\pi_1(B)$ acts trivially on $H^*(F;\mathbb{Q})$ then

$$\sigma(E) = \sigma(F)\sigma(B) \in \mathbb{Z}$$ *(using spectral sequences)*

The non-multiplicativity of the signature

WARNING!

In general the signature of a fibre bundle is NOT multiplicative:

$$\sigma(E) \neq \sigma(F)\sigma(B) \in \mathbb{Z}$$

Example

Kodaira 1967 and Atiyah 1969, constructed non-multiplicative examples of fibre bundles $F \to E \to B$ with $\pi_1(B)$ acting non-trivially on $H^*(F;\mathbb{Q})$: The total space E is a 4-manifold which arises as a complex algebraic surface, and B and F are compact oriented surfaces of genus 129 and 6 respectively,

$$\sigma(E) = 2^8 \neq \sigma(F)\sigma(B) = 0 \in \mathbb{Z} \quad \textit{(using index theory)}$$

The non-multiplicativity of the signature

WARNING!

In general the signature of a fibre bundle is NOT multiplicative:

$$\sigma(E) \neq \sigma(F)\sigma(B) \in \mathbb{Z}$$

Example

Kodaira 1967 and Atiyah 1969, constructed non-multiplicative examples of fibre bundles $F \to E \to B$ with $\pi_1(B)$ acting non-trivially on $H^*(F;\mathbb{Q})$: The total space E is a 4-manifold which arises as a complex algebraic surface, and B and F are compact oriented surfaces of genus 129 and 6 respectively,

$$\sigma(E) = 2^8 \neq \sigma(F)\sigma(B) = 0 \in \mathbb{Z} \quad \textit{(using index theory)}$$

Multiplicativity mod 4 for surface bundles. (Meyer, 1973)

Theorem If $F^2 \to E^4 \to B^2$ is a surface bundle then

$$\sigma(E) \equiv \sigma(F)\sigma(B) = 0 \pmod 4$$ *(using group cohomology)*

For any fibre bundle $F \to E \to B$, Lück and Ranicki (1992) described the symmetric Poincaré structure of the total space $C(E)$ in terms of the symmetric Poincaré structures on $C(B)$, $C(F)$ and the action of $\pi_1(B)$ on $C(F)$.

Multiplicativity mod 4 for any bundle. (Hambleton, Korzeniewski, Ranicki, 2007)

Theorem If $F^n \to E^{n+m} \to B^m$ is any fibre bundle then

$$\sigma(E) \equiv \sigma(F)\sigma(B) \pmod 4$$ *(using algebraic K-theory)*

Multiplicativity mod 4

Multiplicativity mod 4 for surface bundles. (Meyer, 1973)

Theorem If $F^2 \to E^4 \to B^2$ is a surface bundle then

$$\sigma(E) \equiv \sigma(F)\sigma(B) = 0 \pmod 4 \quad \textit{(using group cohomology)}$$

For any fibre bundle $F \to E \to B$, Lück and Ranicki (1992) described the symmetric Poincaré structure of the total space $C(E)$ in terms of the symmetric Poincaré structures on $C(B)$, $C(F)$ and the action of $\pi_1(B)$ on $C(F)$.

Multiplicativity mod 4 for any bundle. (Hambleton, Korzeniewski, Ranicki, 2007)

Theorem If $F^n \to E^{n+m} \to B^m$ is any fibre bundle then

$$\sigma(E) \equiv \sigma(F)\sigma(B) \pmod 4 \quad \textit{(using algebraic K-theory)}$$

Multiplicativity mod 8

A. Korzeniewski (2005)

Theorem Let $F^{4m} \to E^{4n+4m} \to B^{4n}$ be a fibre bundle such that the action of $\pi_1(B)$ on $(H_{2m}(F;\mathbb{Z})/torsion) \otimes \mathbb{Z}_2$ is trivial then

$$\sigma(E) \equiv \sigma(F)\sigma(B) \pmod 8$$ *(using chain complexes)*

This is a much weaker hypothesis than the one of Chern, Hirzebruch and Serre, with a weaker conclusion.

What happens if we drop the hypothesis completely?

Multiplicativity mod 8

A. Korzeniewski (2005)

Theorem Let $F^{4m} \to E^{4n+4m} \to B^{4n}$ be a fibre bundle such that the action of $\pi_1(B)$ on $(H_{2m}(F;\mathbb{Z})/torsion) \otimes \mathbb{Z}_2$ is trivial then

$$\sigma(E) \equiv \sigma(F)\sigma(B) \pmod 8$$ *(using chain complexes)*

This is a much weaker hypothesis than the one of Chern, Hirzebruch and Serre, with a weaker conclusion.

What happens if we drop the hypothesis completely?

The signature mod 8

Theorem (Morita,1971)

If E is a closed oriented $4k$-dimensional manifold, then

$$\sigma(E) \equiv BK(H^{2k}(E;\mathbb{Z}_2), q) \pmod 8$$

where BK is the Brown-Kervaire $\mathbb{Z}_8$-valued invariant of the $\mathbb{Z}_4$-valued quadratic form q defined by the evaluation of the Pontryagin square,

$$q : H^{2k}(E;\mathbb{Z}_2) \to H^{4k}(E;\mathbb{Z}_4) = \mathbb{Z}_4$$

Methods of proof:

- *spectral sequences (Morita, 1971)*
- *Gauss sums (Taylor, 2001)*
- *Chain complexes and L-theory (Ranicki & R., 2013)*

The signature mod 8

Theorem (Morita,1971)

If E is a closed oriented $4k$-dimensional manifold, then

$$\sigma(E) \equiv BK(H^{2k}(E;\mathbb{Z}_2), q) \pmod 8$$

where BK is the Brown-Kervaire $\mathbb{Z}_8$-valued invariant of the $\mathbb{Z}_4$-valued quadratic form q defined by the evaluation of the Pontryagin square,

$$q : H^{2k}(E;\mathbb{Z}_2) \to H^{4k}(E;\mathbb{Z}_4) = \mathbb{Z}_4$$

Methods of proof:

- *spectral sequences (Morita, 1971)*
- *Gauss sums (Taylor, 2001)*
- *Chain complexes and L-theory (Ranicki & R., 2013)*

New result: the obstruction to multiplicativity mod 8

Hambleton, Korzeniewski and Ranicki proved that for a fibre bundle $F^{4m} \longrightarrow E^{4n+4m} \longrightarrow B^{4n}$

$$\sigma(E) - \sigma(F)\sigma(B) \equiv 0 \pmod 4$$

Theorem (work in progress)

The obstruction to multiplicativity modulo 8 is:

$$\sigma(E) - \sigma(F)\sigma(B) \equiv BK(H_1, q_1) - BK(H_2, q_2) \pmod 8$$

with BK the $\mathbb{Z}_8$-valued Brown-Kervaire invariants of the $\mathbb{Z}_4$-valued quadratic forms (H_1, q_1), (H_2, q_2) defined by the Pontrjagin square, where $H_1 = H^{2n+2m}(E; \mathbb{Z}_2)$ and $H_2 = H^{2n+2m}(F \times B; \mathbb{Z}_2)$.

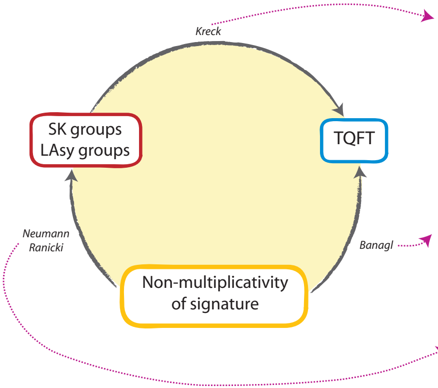
Note that under the hypothesis in Korzeniewski's theorem, the right hand side is zero.

Summary of non-multiplicativity

For fibre bundles of closed, compatibly oriented manifolds:

	multiplicative?	modulo 8?	modulo 4?
$\pi_1(B)$ acts trivially on $H^*(F;\mathbb{Q})$	YES (Chern, Hirzebruch, Serre)	YES	YES
$\pi_1(B)$ acts trivially on $H^{2m}(F;\mathbb{Z})/torsion \otimes \mathbb{Z}_2$	NO	YES (Korzeniewski)	YES
No condition	NO examples by: Kodaira, Atiyah, Hirzebruch	NO Obstruction is $BK(H,q)$	YES (Hambleton, Korzeniewski, Ranicki)

Relations diagram



From a topological point of view, a field theory gives a way to compute an invariant by cutting a manifold into simple pieces and making the function effectively computable. Which invariants of closed manifolds are partition functions of the field theory? No complete answer is known to this quetion, but SK invariants are such partition functions.

Very recently Markus Banagl has described how non-multiplicative fibre bundles give rise to certain TQFTs.

This relationship was described geometrically by Neumann and algebraically in terms of asymmetric L-theory by Ranicki

Purpose of the talk: giving a "signature scent"

"It's less of a spell than it is a signature scent"

Thank you!