27/09/13

Knot invariants from analysis

Patrick Orson

Outline

- Analytic invariants + index theory
- Twisting invariants by representations
- Some knot theory
- Eta invariants of knots.

§1 Analytic invariants (Atiyah-Patodi-Singer)

Let $B: H \to H$ be a bounded linear operator on complex Hilbert space H, $B \in \mathcal{B}(H)$.

B is <u>Fredholm</u> if $\dim \ker B, \dim \ker B^* < \infty$.

<u>Def'n</u> B Fredholm, then

$$\mathrm{index}(B) := \dim \ker B - \dim \ker B^* \in \mathbb{Z}.$$

$\boxed{\text{MAIN EXAMPLE}}$

W a closed $4k$-mfld, recall the deRham cx with cx coeffs:

$$\Omega^r(W) := \Gamma(W, \Lambda^r T^* W \otimes \mathbb{C})$$

$$d: \Omega^{even}(W) \to \Omega^{odd}(W).$$

The Hodge star: $\beta, \alpha \in \Omega^r(W)$

$$* : \Omega^r(W) \to \Omega^{4k-r}(W); \quad \langle \alpha, \beta \rangle = \int_W \alpha \wedge * \beta$$

<u>Def'n</u> The signature operator is

$$B = d + d^* = d - *d* : \Omega^{even}(W) \longrightarrow \Omega^{even}(W)$$

B is a bounded linear operator on the space of L^2-forms, a ex Hilbert space.

<u>Th'm</u> (Hirzebruch signature theorem)

$$\text{index}(B) = \sigma(W)$$

Where $\sigma(W)$ is the "topological signature" of W^{4k} defined by the signature of middle dim int pairing

$$Q_W : H^{2k}(W;\mathbb{R}) \times H^{2k}(W;\mathbb{R}) \longrightarrow \mathbb{R}$$
$$(a, b) \longmapsto \langle a \cup b, [W] \rangle$$
$$= \int_W a \wedge b$$

<u>example</u> Can rewrite $\text{index}(B) = \int_W \mathcal{L}(p_1, \dots, p_k)$ L-genus

Pontrjagin classes $p_i \in H^{4i}(W;\mathbb{R})$

<u>e.g.</u> $k = 1$, W a 4-mfld

curvature matrix

$$\mathcal{L}(p_1) = \frac{1}{3} p_1, \quad p_1 = \frac{1}{(2\pi)^2} \operatorname{Tr}(F^2)$$

$$\Rightarrow \quad \sigma(W) = \int_W \frac{1}{3(2\pi)^2} \operatorname{Tr}(F^2).$$

Now consider a self-adjoint operator $B : H \to H \in \mathcal{B}(H)$.
If B is Fredholm then $\mathrm{index}(B) = 0$.

Are there "secondary invariants" beyond the index?

Note $\mathrm{Spec}(B) = \{ \lambda \in \mathbb{C} \mid B(w) = \lambda \bar{w} \} \subset \mathbb{R}$

by self adjointness.

The insight of Atiyah-Patodi-Singer is that the 'asymetry' of the spectrum around $0 \in \mathbb{R}$ is a secondary invariant of B. Define for $s \in \mathbb{C}$, $\mathrm{Re}(s) \gg 0$

$$\eta_B(s) = \sum_{\lambda \neq 0} \mathrm{sgn}(\lambda) |\lambda|^{-s} \in \mathbb{R}.$$

<u>Th'm</u> (APS) This has mero continuation to all of $\mathbb{C}$ and is finite, real at 0

$$\eta_B := \eta_B(0) \in \mathbb{R}.$$

$\boxed{\text{MAIN EXAMPLE}}$ Let $\partial W^{4k} = M^{4k-1} \neq \emptyset$

(<u>APS condition</u>) <u>Assume</u> W is collared near the boundary.

When we restrict $B = d + d^*$ to the boundary collar $M \times I$, it looks like

$$B : \Omega^{\text{even}}(M) \to \Omega^{\text{even}}(M)$$

$$\varphi \in \Omega^{2r}(M) \Rightarrow B\varphi = i^{k+1}(-1)^{r+1}(*d - d*)\varphi.$$

Self adjoint + bounded

So we can take the eta-invariant of $B|_M$:

$$\eta(\underset{\substack{\uparrow \\ \text{metric}}}{M, g}) := \eta_{B|_M}$$

Theorem (APS)

$$\int_W \mathcal{L}(p_1,\dots,p_k) - \sigma(W) = \eta(M,g)$$

Note that W has boundary so we must define $\sigma(W)$ using Poincaré-Lefschetz duality now. ◻

---

§2 Twisting invariants

Suppose $\alpha : \pi_1(M) \longrightarrow U(k)$ is a rep that extends to $\beta : \pi_1(W) \longrightarrow U(k)$

We say $\partial(W,\beta) = (M,\alpha)$ and define a bordism theory

$$\Omega_{4k-1}(G) = \{\text{closed } G\text{-mflds}\} \Big/ (M,\alpha) = \partial(W,\beta)$$

(where in our case $G = U(k)$).

Then we can "twist" everything by the rep.

- $C_*(\underset{\substack{\text{univ} \\ \text{cover}}}{\widehat{W}}; \mathbb{Z})$ – a chain cx of f.g. free $\mathbb{Z}\pi_1(W)$-modules
  $\Rightarrow$ can twist by β.

$$H^*_\beta(W;\mathbb{C}^k) := H_*\left(\mathrm{Hom}\left(C_*(\tilde{W})\otimes_\beta \mathbb{C}^k, \mathbb{C}\right)\right)$$

Poincaré duality can be equivariantly defined

$\Rightarrow$ equivariant intersection pairing

$$Q^\beta_W : H^*_\beta(W;\mathbb{C}^k)\times H^*_\beta(W;\mathbb{C}^k) \longrightarrow \mathbb{R}$$

and $\sigma_\beta(W)$.

- Twist the eta-invariant by using a flat $\mathbb{C}^k$-bundle over M

$$\mathbb{C}^k \longrightarrow V_\alpha \longrightarrow M$$

V_α is canonically defined by setting the monodromy to be α. The flat connection is also canonically defined.

$$\Rightarrow \Omega^r(M;V_\alpha) := \Gamma(M, \Lambda^r T^*M\otimes V_\alpha)$$

$\Rightarrow$ twisted signature operator B_α

$\Rightarrow$ twisted eta-invariant $\eta_\alpha(M,g)$ or $\eta(M,\alpha,g)$

<u>Th'm</u> (APS) Suppose $\partial(W,\beta) = \bigsqcup_n (M,\alpha)$, then

$$\boxed{\tilde{\eta}_\alpha(M) = \frac{1}{n}\left(\sigma_\beta(W) - k\,\sigma(W)\right)}$$

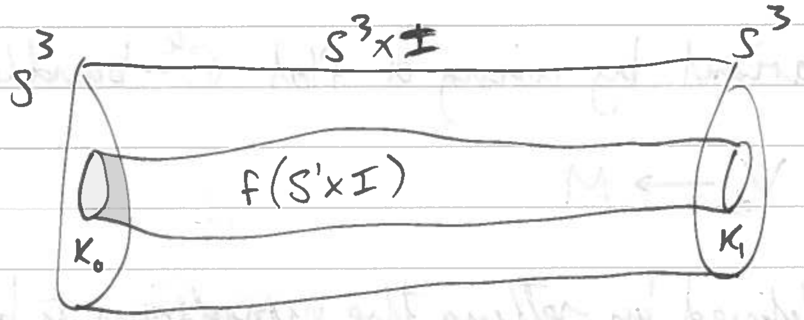
where $\tilde{\eta}_\alpha(M) = \eta(M,\alpha,g) - k\,\eta(M,1,g)$ is ind of g.

§3 Some knot theory

$K: S^1 \hookrightarrow S^3$, $X_K := \overline{S^3 \setminus K \times D^2}$ knot exterior

Q1. Want isotopy inv of knots
Q2. Want knot-cobordism invariants of knots

<u>Def'n</u> A knot-cobordism is a smooth embedding $f: S^1 \times I \hookrightarrow S^3 \times I$ s.t. $f_t := f|_{S^1 \times \{t\}}$ is a knot.



<u>Th'm</u> (?) If there is a homeo $X_K \cong X_{K'}$ then $K \sim K'$ isotopic.

So actually Q1 can be phrased in terms of X_K.

<u>Prop</u> The meridian map $pr_2 : \partial X_K = K \times S^1 \to S^1$ extends to $X_K \to S^1$ and is an iso on homology

$$\varphi_* : H_*(X_K) \xrightarrow{\cong} H_*(S^1)$$

<u>Proof</u> Build the extension cell-by-cell, there are no obstructions. □

<u>Lemma</u> $\pi_1(X_K) \cong \mathbb{Z}$ iff K is the unknot. □

Define the "0-surgery":

$$M_K := X_K \cup (D^2 \times S^1) \qquad \partial D^2 = \text{longitude}$$

<u>FACT</u> The meridian survives this and

$$H_1(X_K) \xrightarrow{\cong} H_1(M_K)$$
$$[m] \longmapsto [m]$$

($\pi_1(X_K) \neq \pi_1(M_K)$ but we can calculate one from the other)

So we now have a <u>closed</u> 3-manifold M with a distinguished map

$$\varphi : \pi_1(M) \longrightarrow \mathbb{Z}$$

that tells us about X and hence K.

<u>§4 Eta-invariants and knots</u>

<u>Example</u> All eta-invariants of the unknot vanish

$$X_K = \overline{S^3 \setminus S \times D^2} = D^2 \times S^1 \Rightarrow M_K = D^2 \times S^1 \underset{\partial D^2 = \partial D^2}{\cup} D^2 \times S^1$$
$$= S^2 \times S^1$$
$$= \partial(D^3 \times S^1)$$
$$!!$$
$$W$$

Suppose $\alpha : \pi_1(M_K) \longrightarrow U(k)$
$\qquad\qquad\quad \cong$
then always $\exists$ $\beta : \pi_1(W)$

$$\left.\begin{array}{r} \text{But} \quad H^2(W) = 0 \\ H^2_\beta(W;\mathbb{C}^k) = 0 \end{array}\right\} \Rightarrow \tilde{\eta}_\alpha(M) = 0$$

What other reps can we come up with?

Recall $\varphi : \pi_1(M_K) \longrightarrow \mathbb{Z}$ the meridian map.
$\qquad\qquad [m] \longmapsto 1$

This defines a $U(1)$ rep for any fixed $\omega \in U(1)$ by sending 1 to ω.

$$\begin{aligned} \alpha_\omega : \pi_1(M_K) &\longrightarrow \mathbb{Z} \longrightarrow U(1) \\ [m] &\longmapsto 1 \longmapsto \omega \end{aligned}$$

i.e. $\alpha_\omega := \omega^{\varphi(-)}$

It is always possible to choose a bounding 4-manifold extending this rep:

(Connor Floyd)
$$\begin{aligned} \Omega_3(G) &\cong \Omega_3 \oplus H_3(G) \\ &= \{e\} \oplus H_3(K(G,1)) \\ &= H_3(K(G,1)) \end{aligned}$$

But $S^1 \simeq K(\mathbb{Z},1) \Rightarrow H_3(\mathbb{Z}) = \Omega_3(\mathbb{Z}) = 0$

$\Rightarrow$ all $\mathbb{Z}$-manifolds are nullbordant in dim 3.

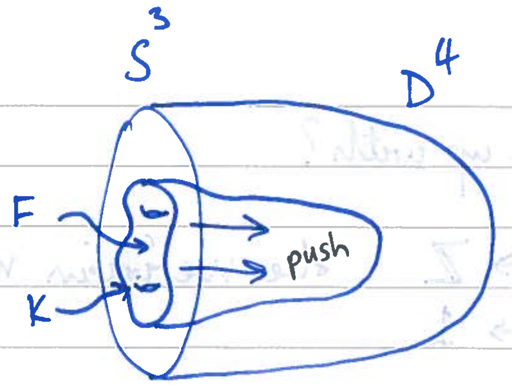
$\Rightarrow \exists\ W^4$ s.t. $\partial W = M$ and $\exists\ \psi : \pi_1(W) \to \mathbb{Z}$

$$\begin{tikzcd} \pi_1(W) \arrow[r] \arrow[dr] & \mathbb{Z} \\ & \pi_1(M) \arrow[u] \end{tikzcd}$$

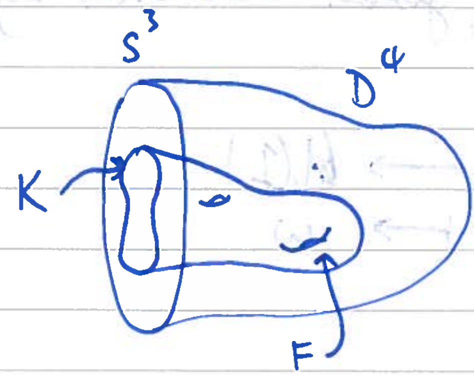
But what is W?

Answer: Can build it using a Seifert surface.

Proof (Litherland)

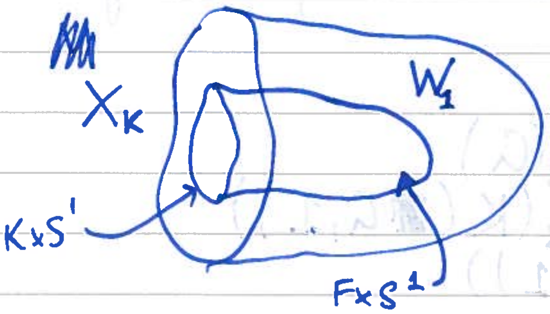


$\Downarrow$



fatten everything up and remove

$\Downarrow$



$$W_1 = \overline{D^4 \setminus F\times D^2}$$
$$X_K = \overline{S^3 \setminus K\times D^2}$$
$$\partial W_1 = X_K \cup_{K\times S^1} (F\times S^1)$$

The map $X_K \xrightarrow{\varphi} \mathbb{Z}$ extends over W_1

Glue in $H\times S^1$ where $\partial H = F\times D^2$

so that $W = W_1 \cup_{F\times\partial D^2} H\times\partial D^2$

and $\partial W = X_K \cup D^2\times S^1 = M_K$

Follow the generators of π_1 and we get $\varphi : \pi_1(M_K) \to \mathbb{Z}$ s.t.

$$\begin{tikzcd} \pi_1(M_K) \arrow[r] \arrow[dr] & \mathbb{Z} \\ & \pi_1(W) \arrow[u] \end{tikzcd}$$

Moreover if you're careful you can show that the intersection matrix of W, and the twisted intersection matrix of W are related by

$$-[Q_W] + [Q_W^\beta] = (1-\omega)V + (1-\bar{\omega})V^*$$

where V is the Seifert matrix for F.

$$\Rightarrow \sigma(LHS) = \sigma(RHS)$$

"Tristram-Levine signature"

and hence

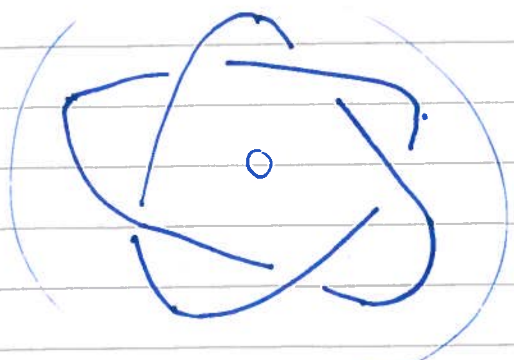
Litherland)

$$\boxed{\eta_{\alpha_\omega}(M) = \sigma\left((1-\omega)V + (1-\bar{\omega})V^*\right)}$$

VERY COMPUTABLE!

<u>example</u> (Julia Collins)

$K = 5_1$ the Cinquefoil knot or $(2,5)$ torus knot.

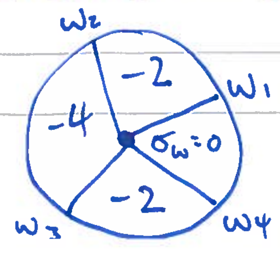


One Seifert matrix (from Mathematica)

$$\begin{pmatrix} -1 & -1 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 \\ 0 & -1 & 0 & -1 \end{pmatrix} = V$$

For $\omega \neq e^{2\pi i x_K}$ where $x_i = \frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}$ we may calculate the signature of $(1-\omega)V + (1-\bar{\omega})V^*$

$\omega =$



<u>Theorem</u> (T-L) ω-sigs are bordism inv away from roots of Alex poly.