Perverse sheaves & Decomposition Thm

Introductory talk

The theory of perverse sheaves originates from intersection homology theory.

$\boxed{\text{fall 1974, IHES}}$ ⟶ (Motivated by a question by Sullivan) Goresky & MacPherson discover intersection homology

AIM: find a "good" topological invt ($\sim$ (co)homology) for singular spaces
having some of the nice properties (co)homology of proj. non-sing. $\mathbb{C}$-var.'s has:

- $\boxed{\text{Poincaré Duality}}$
- Lefschetz hyperplane
- Hodge signature Thm
- (relative) Hard Lefschetz

Rmk: all play a role in Decataldo–Migliorini pf of Decomp. Thm.

Let X be a "sufficiently nice" stratified topological space.

$$\rightsquigarrow \quad IH^{\bar{p}}(X) = \frac{\text{certain cycles}}{\text{certain homologies}} \rightarrow \text{restricted as to how they meet } X^{sing}$$

singular locus of X

$\bar{p}$ = perversity (function from the set of singular strata to $\mathbb{Z}_{(\geq 0)}$), it gives allowability conditions

Why is it call perversity? ⟶ MacPherson, "Intersection Homology and Perverse Sheaves", 1990

" If $\bar{p} = \bar{0}$, the cycles and their boundaries dip into the singularities as little as they can be expected. As the perversity grows, this transversality condition is relaxed, and they're allowed to go deeper into the singularities. If you regard transversality as the nicest behaviour that a cycle could have wrt a singularity, then lack of transversality can be thought as "perverse". "

More on the history of IH:

- S.L. Kleinman, "The Development of Intersection Homology Theory"

Deligne (& Verdier): move from groups to sheaves! (complexes of)

$IH_\bullet^{\bar{p}}(X)$ = total homology gps of a cplx $IC_\bullet^{\bar{p}}(X) \subset C_\bullet(X)$ ← ordinary chains

defining conditions of $IC_\bullet^{\bar{p}}(X)$ are local

$\Downarrow$

can consider $IC_\bullet^{\bar{p}}(U)$ for any $U \underset{\text{open}}{\subset} X$ ⇝ sheaf.

Sheaf theoretic approach has several advantages: (cf [GM2])

(1) $IC_\bullet$ is now living in $\mathcal{D}^b(X)$ ⟶ functorial machinery

← bounded derived cat. of sheaves on X

(2) can give an alternative realisation of IC by using standard operations of sheaf theory

↓ Deligne construction

(3) stratification free characterisation of $IC^\bullet$

Usual cohomology: $H^\bullet(X, k) = H^\bullet(X, \underline{k}_X)$ ← constant sheaf

VS

Intersection cohomology: $IH^\bullet(X, k) = H^\bullet(X, IC(X))$

If X irreducible cplx alg. variety
$X_0 \subset X$ open dense, smooth subvariety ⟶ $IC^\bullet(X)$ = certain "canonical" extension of $\underline{k}_{X_0}$

in general, not as a sheaf, but as a complex of sheaves (well-defined up to q. iso)

Applications in Rep Theory

Intersection cohomology methods allows to approach representation theoretic problems geometrically:

representation category $\longleftrightarrow$ category of perverse sheaves on a suitable stratified space

Let G be a conn'd reductive alg. gp $/\mathbb{C}$

it's a non singular variety but studying rep theory of $\to G$, $\to \mathrm{Lie}\, G = \mathfrak{g}$, $\to G((t))$, ... $\rightsquigarrow$ singular varieties.

Some examples: (char $=0$)

1) KL-Theory $\quad \mathcal{O}_0$ of $\mathfrak{g} \longleftrightarrow \mathrm{Perv}(G/B)$ flag variety

$B \curvearrowright G/B = \bigsqcup BwB/B \quad \overline{BwB/B}$ singular (in general)

2) Springer corresp. $\quad \mathrm{Irr}\, \mathbb{C}W \hookrightarrow \mathrm{Perv}(\mathcal{N})$ nilpotent cone $\subset \mathfrak{g}$

$G \curvearrowright \mathcal{N} = \bigsqcup \mathcal{O}$

3) Geom. Satake $\quad (\mathrm{Rep}(G^\vee), \otimes) \longleftrightarrow \mathrm{Perv}\left(G((z))/G[[z]], *\right)$ affine Grassmannian, conv. product

$G[[z]] \curvearrowright G((z))/G[[z]] = \bigsqcup \mathcal{O}$

Why are perverse sheaves so powerful in rep theory?

One of the reasons is: Decomposition Thm! (Beilinson-Bernstein-Deligne(-Gabber))

DT: $\left(\bigoplus IC(X_w)[m]\right)$ preserved under proper pushforward

semisimple complexes

$\Longrightarrow$ inductive calculation of (stalks) of $IC^\cdot$

$\Longrightarrow$ (for example) KL-conjecture.